Durham University

EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
|---|---|---|
| May/June | 2023 | MATH3111-WE01 |

**Title:**

Quantum Mechanics III

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| Time: | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

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| Instructions to Candidates: | Answer all questions.<br>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.<br>Students must use the mathematics specific answer book. |

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| **Revision:** | |

ED01/2023

## SECTION A

**Q1** Consider a two dimensional Hilbert space on which the physical observable $\hat{A}$ and the state $|\psi\rangle$ take the matrix form

$$\hat{A} = \begin{pmatrix} 1 & x \\ 2+i & 3 \end{pmatrix}, \quad |\psi\rangle = \begin{pmatrix} 1/2 \\ c \end{pmatrix} .$$

(a) Find $x$ and $c$ where $c$ is positive and real.

(b) Find the possible outcomes of a measurement of $\hat{A}$.

(c) Calculate the expectation value $\langle A \rangle$ for the state $|\psi\rangle$.

**Q2** Consider a unitary operator $\hat{U}$ with non-degenerate spectrum in an $n$-dimensional Hilbert space. Determine whether the following statements are true or false, and provide a proof or a counter-example as appropriate.

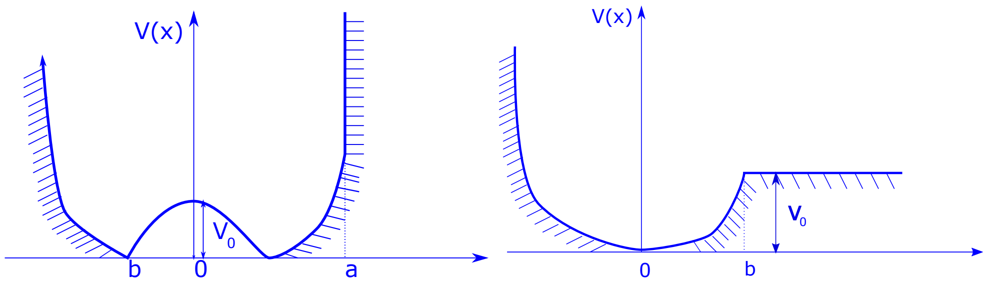
(a) The operator $\hat{U}^k$ is a unitary operator for any positive integer $k$.

(b) The determinant of $\hat{U}$ is a complex number of unit modulus.

(c) The eigenvalues of $\hat{U}$ are complex numbers of unit modulus.

(d) The eigenvectors of $\hat{U}$ are mutually orthogonal.

**Q3** A quantum particle propagating in one of the following two one-dimensional potentials, shown on the image. In the first figure at $x = a$ the potential is infinite and as $x \to -\infty$,$V(x) \to \infty$. In the second figure, for $x > b$ potential is constant, with value $V(x) = V_0 > 0$, and also as $x \to -\infty$, $V(x) \to \infty$.




Answer the following questions:

(a) For each of these potentials, explain whether the energy spectrum of the particle is discrete or continuous? If your answer depends on the energy of the particle, please explain and state which part of the spectrum is discrete and which continuous.

(b) Sketch the qualitative behaviour of the wave function for the semi-classical particle in both potentials and in particular state what is the qualitative behaviour of the amplitudes and frequencies in various regions of space.

(c) If a particle is created in the first potential somewhere in the region $b < x < a$ with energy $E > V_0$ what is the probability for this particle to escape to $-\infty$ and why?

(d) If a beam of particles is shot from $x \to +\infty$, in the second potential, what is the reflection coefficient? Explain and argue your reasoning.

**Q4** A quantum particle has angular momentum $J$.

(a) What are the allowed values of $J$ in quantum theory? Write the Hilbert space of allowed states which this particle can have. Explain all the quantum numbers which appear in each state.

(b) Work out a matrix representation of the raising/lowering operator $\hat{L}_\pm$ in the Hilbert space of this particle.

(c) Evaluate the following commutators

$$[\hat{x}, \hat{L}_y^2]\,, \qquad [\hat{L}_z, \hat{L}_\pm]$$

(d) Evaluate the expectation value of the commutator $[\hat{L}_z, \hat{L}_\pm]$ in an arbitrary eigen-state of $\hat{L}^2$ and $\hat{L}_z$ .

## SECTION B

**Q5** (a) If $\hat{H}$ is the Hamiltonian of a quantum system and $\hat{A}$ a physical observable which does not depend on time explicitly, state the necessary and sufficient condition under which the expectation value $\langle \hat{A} \rangle$ is independent of time for all quantum states.

(b) If the Hilbert space of the previous system is two dimensional and

$$\hat{H} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} a & i \\ b & 2 \end{pmatrix}$$

find the values of the complex numbers $a$ and $b$ which make $\langle \hat{A} \rangle$ both an observable and a conserved quantity.

(c) Suppose we instead consider an operator $\hat{B}$ which is diagonal and of the form

$$\hat{B} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} .$$

At $t = 0$ we take a measurement for $\hat{B}$ and we find the value $c$. After that, we let the state $|\psi\rangle$ evolve. Find the time evolution of the state, and the expectation value $\langle \hat{B} \rangle$ as a function of time.

(d) Hence or otherwise determine the earliest time that we are certain a measurement of $\hat{B}$ will yield the value $d$.

**Q6** Eigenstates of the linear harmonic oscillator are generated by operators $\hat{a}$ and $\hat{a}^\dagger$ defined as

$$\hat{a} = \sqrt{\frac{1}{2m\omega\hbar}}\left(m\omega\hat{x} + i\hat{p}\right), \quad \hat{a}^\dagger = \sqrt{\frac{1}{2m\omega\hbar}}\left(m\omega\hat{x} - i\hat{p}\right) .$$

The $n$th energy eigenstate is

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$$

where $|0\rangle$ is the ground state (i.e. $\hat{a}|0\rangle = 0$).

(a) Using the commutation relation $[\hat{x}, \hat{p}] = i\hbar$, where $\hbar$ is the reduced Planck's constant, express the Hamiltonian of the linear harmonic oscillator

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

in terms of the number operator, $\hat{N} = \hat{a}^\dagger\hat{a}$.

(b) Show that

$$[\hat{a}, (\hat{a}^\dagger)^n] = n(\hat{a}^\dagger)^{n-1} , \quad [\hat{a}^n, \hat{a}^\dagger] = n\hat{a}^{n-1} .$$

(c) Hence or otherwise evaluate the energy of the linear harmonic oscillator in the state $|n\rangle$.

(d) Evaluate the expectation values $\langle n|\hat{p}^2|n\rangle$ and $\langle n|\hat{x}^2|n\rangle$.

(e) Hence or otherwise show that the ground state of the simple harmonic oscillator, $|0\rangle$, is the only energy eigenstate that saturates the Heisenberg uncertainty inequality, $\Delta x \Delta p \geq \frac{\hbar}{2}$, where

$$\Delta\hat{O} = \sqrt{\langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2} .$$

**Q7** A quantum particle is propagating in three dimensions in the presence of the quadratic potential $V(r, \varphi, \theta) = r^2$. The Laplacian in spherical coordinates is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{r^2}\,.$$

Answer the following questions:

(a) Write the ansatz for the wave function of this quantum particle, and by separation of variables, work out the radial and angular equations. What are the solutions of the angular equation?

(b) By changing variables, from the radial part of the wave function $R(r)$ to $u(r)$ via $u(r) = rR(r)$, simplify the radial equation and extract from it the expression for the effective potential $V_{eff}(r)$.

(c) Work out the behaviour of $V_{eff}$ as $r \to 0$ and as $r \to \infty$ and determine whether this potential has extrema. Using all this information, sketch the graph for the effective potential.

(d) Apply the WKB approximation to the radial equation and write down the quantisation condition. You do not need to integrate this condition explicitly, or write down an analytic expressions for the bounds of the integral, but you need to write down the equations which fix the bounds of the integral. Apply the WKB approximation to compute explicitly what is the energy of the ground state. Explain whether you think that the WKB approximation should or should not be used to evaluate the ground state energy.

(e) By taking into account both information from the radial and angular parts, explain what are the quantum numbers which specify the energy eigen-states of the quantum particle, explain whether the spectrum is degenerate or not, and if it is, what is the degeneracy of each state.

**HINT:** You may find the following integral useful

$$\int \sqrt{b^2 - x^2} = \frac{x\sqrt{b^2 - x^2}}{2} + \frac{b^2}{2}\arcsin\left(\frac{x}{b}\right)$$

**Q8** A quantum particle is located in a space $(x, \varphi)$, where $\varphi \in [0, 2\pi]$ is a periodic coordinate, while $x \in [0, H]$ is a non-periodic coordinate. The particle is also subject to a pertubation,

$$\delta V(x, \varphi) = \begin{cases} V_0 & 0 \leq x \leq H \quad \text{and} \quad 0 \leq \varphi \leq \varphi_0 \\ 0 & 0 \leq x \leq H \quad \text{and} \quad \varphi > \varphi_0 \end{cases}$$

so this potential is non-vanishing in a 'strip' of width $\varphi_0$ and height $H$.

Answer the following questions:

(a) First ignore the potential $\delta V(x, \varphi)$ and write down the Schrödinger equation for the free particle. By using the appropriate ansatz, and by separation of variables, rewrite the Schrödinger equation as a system of two second-order equations.

(b) Write down the general solutions to these two equations and impose appropriate boundary conditions which these solutions have to satisfy.

(c) Explicitly write down the spectrum of the unperturbed system, i.e. write down the properly normalised energy eigen-states as well as their energies and all the quantum numbers for these states. Is the spectrum degenerate or not? Please explain your answer.

(d) Now assume that a potential $\delta V(x, \varphi)$ is turned on and that $H = 1/\sqrt{2}$. By explicit computation find the first-order correction to the energy of the ground state and first excited state. For the first excited state you do not need to solve the quadratic equation which fixes the corrections to the energy.

**HINT:** You can think about the restriction in the $x$-direction as if the particle is on line and there is an infinite potential at $x = 0$ and $x = H$, and also the Laplacian on this surface is just standard: $(\partial/\partial\varphi)^2 + (\partial/\partial x)^2$.