Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3141-WE01

Title:
Operations Research III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

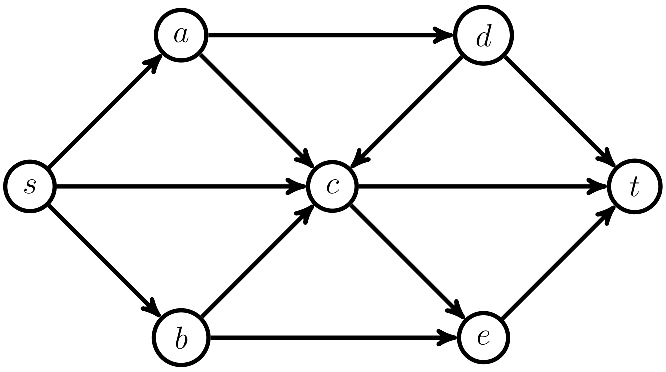
ED01/2023

SECTION A

Q1 Find the optimal value of the following linear program. State all feasible values of x_1, x_2, and x_3 for which the optimal value is attained.

$$\begin{aligned} \max\ & 3x_1 - x_2 + 4x_3 \\ \text{subject to}\ & x_1 + 2x_2 + 3x_3 \geq 1 \\ & x_1 + x_2 + x_3 \leq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_3 \geq 0 \end{aligned}$$

Q2 **2.1** Consider the following directed graph:



A collection of directed paths is called *edge-disjoint* if no paths in the collection have any edges in common. For instance, $\{(s, a, c, t), (s, c, e, t)\}$ is edge-disjoint, but $\{(s, a, c, t), (s, c, t)\}$ is not. Intuitively and without proof,

- identify an as large as possible (i.e. containing as many paths as possible) edge-disjoint collection of directed paths from s to t, and
- identify an as small as possible (i.e. containing as few edges as possible) cut separating s from t, directed from s to t.

2.2 Now consider any arbitrary directed graph, and any two vertices on the graph, labelled s (for source) and t (for terminus). Show that the size (i.e. number of edges) of the smallest cut separating s from t and directed from s to t, is equal to the size (i.e. number of paths) of the largest edge-disjoint collection of directed paths from s to t. You may use any results from the lectures as long as they are clearly stated, and as long as you make it clear how exactly you are applying those results.

Q3 An organic farmer needs to decide whether to plant crops each year, for the next 3 years. The state of his field in the next year depends on the state in the current year, as well as whether any crops are planted in the current year. The field can be in a good ($i = 2$), fair ($i = 1$), or poor ($i = 0$) state. Transition probabilities are as follows if crops are planted (say $x = 1$), where i denotes the current state and j denotes the next state:

$$\begin{array}{lll} P(j=2 \mid i=2) = 0.3 & P(j=1 \mid i=2) = 0.5 & P(j=0 \mid i=2) = 0.2 \\ P(j=2 \mid i=1) = 0 & P(j=1 \mid i=1) = 0.4 & P(j=0 \mid i=1) = 0.6 \\ P(j=2 \mid i=0) = 0 & P(j=1 \mid i=0) = 0 & P(j=0 \mid i=0) = 1 \end{array}$$

If no crops are planted (say $x = 0$), the field can recover, and the transition probabilities are in that case:

$$\begin{array}{lll} P(j=2 \mid i=2) = 1 & P(j=1 \mid i=2) = 0 & P(j=0 \mid i=2) = 0 \\ P(j=2 \mid i=1) = 0.8 & P(j=1 \mid i=1) = 0.2 & P(j=0 \mid i=1) = 0 \\ P(j=2 \mid i=0) = 0.2 & P(j=1 \mid i=0) = 0.8 & P(j=0 \mid i=0) = 0 \end{array}$$

Planting crops gives a return of 9 (in units of £100,000) if the current state is good, 5 if the current state is fair, and 2 if the current state is poor. The return is 0 if no crops are planted. Initially, the field is in a fair state.

3.1 Formulate the problem of maximizing the total monetary gain over the next 3 years as a stochastic dynamic programming problem.

3.2 Use stochastic dynamic programming to find all planting policies that maximize the total monetary gain over the next 3 years.

Q4 **4.1** State the condition under which a Markov chain has a unique stationary distribution. Briefly explain how you can verify this condition graphically.

4.2 On any given day, weather can be sunny, cloudy, or rainy. Zack, an icecream vendor, knows that, on average, he sells 150 icecreams on a sunny day, 100 on a cloudy day, but only 30 on a rainy day. From past weather data, he obtained a Markov chain model for the weather at his icecream stand with the following transition matrix, where state 1 is sunny, state 2 is cloudy, and state 3 is rainy:

$$P = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.5 & 0 & 0.5 \\ 0.3 & 0.7 & 0 \end{bmatrix}$$

Calculate the long-run average number of icecreams sold per day. You may use any results from the lectures, as long as they are stated clearly.

SECTION B

Q5 An humanitarian aid agency needs to transport its goods from various supporting countries to two countries in need. Transportation costs, supplies, and demands, are respectively given by:

$$[c_{ij}] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 2 \\ 2 & 3 \end{bmatrix} \qquad [a_i] = \begin{bmatrix} 30 \\ 20 \\ 20 \\ 40 \end{bmatrix} \qquad [b_j] = \begin{bmatrix} 40 & 70 \end{bmatrix}$$

5.1 Find all optimal transportation schemes.

5.2 Due to a natural disaster, supporting country 1 (with supply 30) can no longer supply aid. Again, find all optimal transportation schemes, assuming unmet demand induces a cost of 5.

Q6 To maintain an optimal fishing population in a certain region, the government wants to re-evaluate its yearly fishing policy which depends on measurements of the fishing stock at the start of each year. Each year, the possible decisions are to fish ($k = F$), or not to fish ($k = N$). The fishing stock can be either high (state 1), medium (state 2), or low (state 3). The transition matrix $P(k)$ between states under each action k is:

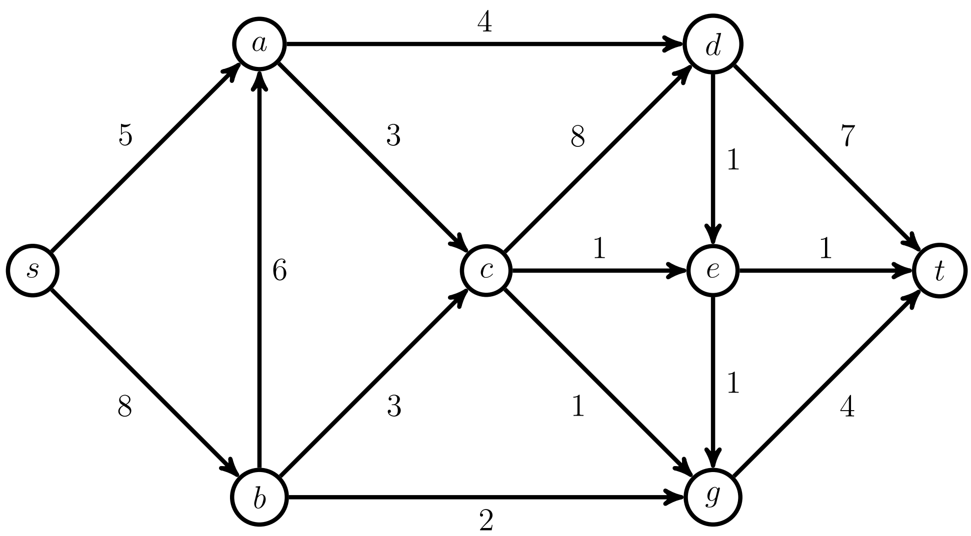
$$P(F) = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0 & 0.1 & 0.9 \\ 0 & 0 & 1 \end{bmatrix} \qquad P(N) = \begin{bmatrix} 1 & 0 & 0 \\ 0.9 & 0.1 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

Fishing earns 10 (in millions of pounds) from a high stock, 4 from a medium stock, and 1 from a low stock. No fishing costs 2, independently of the fishing stock.

6.1 The current policy is to fish if the stock is high or medium, and not to fish if the stock is low. Is this policy optimal with respect to maximizing long-run average profit? If not, can you find a better policy?

6.2 Suppose instead the government would like to maximize the long-run average chance of having a medium fishing stock, regardless of economic income or cost. How would this change the mathematical formulation of the problem? Consider the policy ‘fish if high, and do not fish otherwise’. Is this policy optimal with respect to the new objective?

Q7 Consider a traffic system that can be represented by the following flow network, with capacities indicated on each arc:



7.1 Use the Ford-Fulkerson labelling algorithm to identify a maximum flow from the source s to the terminus t.

7.2 Funding is available to increase the capacity along one of the arcs. Can the maximal flow be increased by increasing the capacity along just one arc? If yes, identify such arc and find the new optimal flow. If no, provide a theoretical argument (you may use, without proof, any theorems proved in the lectures, as long as you state them clearly).

Q8 Consider the following optimization problem (noting the non-linear constraint):

$$\text{maximize} \quad \min\{x_1^2, 12\} + 2x_2 + 3x_3 \quad \text{subject to} \quad 3 \leq x_1x_2 + x_3 \leq 6$$

with $x_i \in \{1, 2, 3, 4\}$ for all $i \in \{1, 2, 3\}$.

8.1 Formulate this problem as a dynamic programming problem.

8.2 Solve the problem using dynamic programming: identify the optimal value, as well as all optimal solutions for x_1, x_2, and x_3.