

## **EXAMINATION PAPER**

Examination Session: May/June

2023

Year:

Exam Code:

MATH3201-WE01

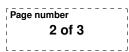
Title:

Geometry III

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions.
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.
	Students must use the mathematics specific answer book.

**Revision:** 

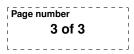


## SECTION A

- **Q1** Let r be the reflection on the Euclidean plane with respect to the line x = 0. Let R be the anticlockwise rotation about the origin O = (0, 0) though the angle  $\pi/2$ .
  - (a) What is the type of the transformation  $\psi = R \circ r \circ R^{-1}$ ?
  - (b) Find the fixed points of the transformation  $\psi$  described in part (a).
- Q2 (a) Is it true or false that affine transformations act transitively on quadrilaterals in the Euclidean plane? Justify your answer.
  - (b) A hexagon ABCDEF in the Euclidean plane is symmetric with respect to the diagonal AD. Is it always true that there exists a projective map taking ABCDEF to a regular hexagon? Justify your answer.
- **Q3** Let *ABCDE* be a hyperbolic pentagon with AB = BC = CD = DE = a and  $\angle ABC = \angle BCD = \angle CDE = \pi/2$ .
  - (a) Let  $\gamma = \angle BCA$ . Express  $\sin \gamma$  in terms of a.
  - (b) Express the length of AE in terms of a.
- **Q4** Let  $\gamma$  be a circle of radius  $0 < r < \pi/2$  on the unit sphere  $S^2$ . Let AB be a diameter of  $\gamma$  and let C be a point on  $\gamma$  distinct from A and B.
  - (a) Show that, unlike the Euclidean case,  $\angle ACB$  is not necessarily equal to  $\pi/2$ .
  - (b) Is it true that the size of  $\angle ACB$  does not depend on the choice of  $C \in \gamma$ ? Justify your answer.

## SECTION B

- **Q5** (a) Does there exist a Möbius transformation taking the points -1, 0, 1 + i, 2 + i to the points 5, 4 + 3i, -3 + 4i, -4 3i respectively? Justify your answer.
  - (b) Let  $\gamma_1$  and  $\gamma_2$  be two circles with centres  $O_1$  and  $O_2$  respectively on the Euclidean plane. Let f be a Möbius transformation taking  $\gamma_1$  to  $\gamma_2$ . Is it always true that  $f(O_1) = O_2$ ? Justify your answer.
  - (c) Consider the four circles of radius 1 centred at the points 1+i, -1+i, -1-i, 1-i. How many different Möbius transformations take the union of the four circles to itself (not necessarily pointwise)? Justify your answer.
- Q6 (a) Show that there exists a regular hyperbolic quadrilateral with all angles equal to  $\pi/3$ .
  - (b) Let ABCD be a regular hyperbolic quadrilateral with all angles equal to  $\pi/3$ labelled in the clockwise direction. For every  $X \in \mathbb{H}^2$  denote by  $R_{\pi/3,X}$  a rotation about X through the angle  $\pi/3$  (in anti-clockwise direction). Denote  $f = R_{\pi/3,D} \circ R_{\pi/3,C} \circ R_{\pi/3,B} \circ R_{\pi/3,A}$ . Find the type of the isometry f.
  - (c) Let M be the midpoint of AD. Find  $(R_{\pi/3,B} \circ R_{\pi/3,A})^{2023}(M)$ .



- Q7 (a) Let  $A_1A_2A_3A_4$  be a quadrilateral on the Euclidean plane. Let  $B_i$ , i = 1, ..., 4, be a midpoint of  $A_iA_{i+1}$  (where  $A_5 = A_1$ ). Show that  $B_1B_2 = B_3B_4$  and  $B_2B_3 = B_1B_4$ .
  - (b) Show that the statement of (a) does not hold for a spherical quadrilateral  $A_1A_2A_3A_4$ .
  - (c) Consider the spherical quadrilateral  $A_1A_2A_3A_4$  and the corresponding quadrilateral  $B_1B_2B_3B_4$  constructed as in part (a). Let  $S_{B_1B_2B_3B_4}$  and  $S_{A_1A_2A_3A_4}$  be the areas of the two quadrilaterals. Is it true that  $S_{B_1B_2B_3B_4} = \frac{1}{2}S_{A_1A_2A_3A_4}$ ? Justify your answer.
- **Q8** (a) Find the cross-ratio of the following four points in  $\mathbb{R}P^2$  (given in homogeneous coordinates):

 $A = (1:0:0), \quad B = (1:1:1), \quad C = (0:1:1), \quad D = (-2:1:1).$ 

- (b) Let  $\triangle A_1 A_2 A_3$  be a triangle and T be a point on the Euclidean plane. Assume that  $T \notin A_i A_j$  for  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ . Let  $B_i = A_i T \cap A_j A_k$  for i = 1, 2, 3 and  $i, j, k \in \{1, 2, 3\}$  distinct indices. Let  $C_i = A_j A_k \cap B_j B_k$  for i = 1, 2, 3 and  $i, j, k \in \{1, 2, 3\}$  distinct indices. Assuming that all the points listed above are distinct and exist, prove that the points  $C_1, C_2, C_3$  are collinear.
- (c) Formulate the statement dual to the one given in part (b).