Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3201-WE01

Title: Geometry III

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Let r be the reflection on the Euclidean plane with respect to the line $x = 0$. Let R be the anticlockwise rotation about the origin $O = (0, 0)$ though the angle $\pi/2$.

(a) What is the type of the transformation $\psi = R \circ r \circ R^{-1}$?

(b) Find the fixed points of the transformation ψ described in part (a).

Q2 (a) Is it true or false that affine transformations act transitively on quadrilaterals in the Euclidean plane? Justify your answer.

(b) A hexagon $ABCDEF$ in the Euclidean plane is symmetric with respect to the diagonal AD. Is it always true that there exists a projective map taking $ABCDEF$ to a regular hexagon? Justify your answer.

Q3 Let $ABCDE$ be a hyperbolic pentagon with $AB = BC = CD = DE = a$ and $\angle ABC = \angle BCD = \angle CDE = \pi/2$.

(a) Let $\gamma = \angle BCA$. Express $\sin\gamma$ in terms of a.

(b) Express the length of AE in terms of a.

Q4 Let γ be a circle of radius $0 < r < \pi/2$ on the unit sphere S^2. Let AB be a diameter of γ and let C be a point on γ distinct from A and B.

(a) Show that, unlike the Euclidean case, $\angle ACB$ is not necessarily equal to $\pi/2$.

(b) Is it true that the size of $\angle ACB$ does not depend on the choice of $C \in \gamma$? Justify your answer.

SECTION B

Q5 (a) Does there exist a Möbius transformation taking the points $-1, 0, 1+i, 2+i$ to the points $5, 4+3i, -3+4i, -4-3i$ respectively? Justify your answer.

(b) Let γ_1 and γ_2 be two circles with centres O_1 and O_2 respectively on the Euclidean plane. Let f be a Möbius transformation taking γ_1 to γ_2. Is it always true that $f(O_1) = O_2$? Justify your answer.

(c) Consider the four circles of radius 1 centred at the points $1+i, -1+i, -1-i, 1-i$. How many different Möbius transformations take the union of the four circles to itself (not necessarily pointwise)? Justify your answer.

Q6 (a) Show that there exists a regular hyperbolic quadrilateral with all angles equal to $\pi/3$.

(b) Let $ABCD$ be a regular hyperbolic quadrilateral with all angles equal to $\pi/3$ labelled in the clockwise direction. For every $X \in \mathbb{H}^2$ denote by $R_{\pi/3,X}$ a rotation about X through the angle $\pi/3$ (in anti-clockwise direction). Denote $f = R_{\pi/3,D} \circ R_{\pi/3,C} \circ R_{\pi/3,B} \circ R_{\pi/3,A}$. Find the type of the isometry f.

(c) Let M be the midpoint of AD. Find $(R_{\pi/3,B} \circ R_{\pi/3,A})^{2023}(M)$.

Q7 (a) Let $A_1A_2A_3A_4$ be a quadrilateral on the Euclidean plane. Let B_i, $i = 1, \dots, 4$, be a midpoint of A_iA_{i+1} (where $A_5 = A_1$). Show that $B_1B_2 = B_3B_4$ and $B_2B_3 = B_1B_4$.

(b) Show that the statement of (a) does not hold for a spherical quadrilateral $A_1A_2A_3A_4$.

(c) Consider the spherical quadrilateral $A_1A_2A_3A_4$ and the corresponding quadrilateral $B_1B_2B_3B_4$ constructed as in part (a). Let $S_{B_1B_2B_3B_4}$ and $S_{A_1A_2A_3A_4}$ be the areas of the two quadrilaterals. Is it true that $S_{B_1B_2B_3B_4} = \frac{1}{2}S_{A_1A_2A_3A_4}$? Justify your answer.

Q8 (a) Find the cross-ratio of the following four points in $\mathbb{R}P^2$ (given in homogeneous coordinates):

$$A = (1:0:0), \quad B = (1:1:1), \quad C = (0:1:1), \quad D = (-2:1:1).$$

(b) Let $\triangle A_1A_2A_3$ be a triangle and T be a point on the Euclidean plane. Assume that $T \notin A_iA_j$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. Let $B_i = A_iT \cap A_jA_k$ for $i = 1, 2, 3$ and $i, j, k \in \{1, 2, 3\}$ distinct indices. Let $C_i = A_jA_k \cap B_jB_k$ for $i = 1, 2, 3$ and $i, j, k \in \{1, 2, 3\}$ distinct indices. Assuming that all the points listed above are distinct and exist, prove that the points C_1, C_2, C_3 are collinear.

(c) Formulate the statement dual to the one given in part (b).