EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3231-WE01

Title:
Solitons III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Show that for each of the following two equations and boundary conditions, the quantity $\int_{-\infty}^{+\infty} dx\ \rho(u, u_x, u_t)$ is constant:

(a) $u_t + u^3 u_x + u_{xxx} = 0$, with $u, u_x, u_{xx} \to 0$ as $x \to \pm\infty$, and $\rho = u^2$;

(b) $u_{tt} - u_{xx} + \sin u = 0$, with $u, u_x, u_t \to 0$ as $x \to -\infty$, $u \to -4\pi, u_x, u_t \to 0$ as $x \to +\infty$, and $\rho = u_t u_x$.

Q2 Show that if $f(x)$ is a smooth function such that $f \to \pm 1$ as $x \to \pm\infty$, then the quantity

$$E = \int_{-\infty}^{+\infty} dx\ \left[(f')^2 + (f^2 - 1)^2\right]$$

satisfies the inequality $E \geq C$, for a positive constant C that you should find. Find a function $f(x)$ for which E equals the minimum value C.

Q3 (a) If $F[u]$ is the functional $F[u] = \int_{-\infty}^{\infty} dx\, f(u, u_x, u_{xx}, \ldots)$, define the functional derivative $\delta F[u]/\delta u$ and derive an expression for this derivative in terms of $\partial f/\partial u$, $\partial f/\partial u_x$, $\partial f/\partial u_{xx}$ etc.

You may assume that u *satisfies the boundary conditions* $u \to 0$, $u_x \to 0$, $u_{xx} \to 0$ *etc as* $x \to \pm\infty$.

(b) Consider $f(u, u_x) = a_1 u_x^2 + a_2 u^3$ and write out the differential equation

$$u_t = \frac{\partial}{\partial x}\left(\frac{\delta F[u]}{\delta u}\right).$$

Find the values of the constants a_1 and a_2 for which this becomes the KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

Q4 The Marchenko equation is

$$K(x, z) + F(x + z) + \int_{-\infty}^{x} dy\, K(x, y)\, F(y + z) = 0$$

Given $F(x) = -e^{-x}\theta(x)$, show that $K(x, z) = \alpha\theta(x + z)$ solves the Marchenko equation for $z \leq x$ for some constant α that you should determine. Hence find the potential $V(x) = 2dK(x, x)/dx$.

[The function $\theta(x)$ *is the Heaviside step function defined as* $\theta(x) = 1$ *for* $x > 0$ *and* $\theta(x) = 0$ *for* $x \leq 0$*. Its derivative is the Dirac delta function:* $d\theta(x)/dx = \delta(x)$*.]*

SECTION B

Q5 Consider the pair of equations

$$u_x = -uv_x - e^v$$
$$u_t = uv_t - e^{-v} \; .$$

Show that these relations give a Bäcklund transform between an equation for u and an equation for v, that you should find.

Q6 Hirota's bilinear differential operators $D_x^m D_t^n$ are defined by

$$\left(D_x^m D_t^n (f,g)\right)(x,t) := (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x',t')\big|_{x'=x,\ t'=t} \; .$$

By setting $u = \log f$ in the differential equation

$$u_{xt} = e^{-u} - e^{-2u} \; ,$$

show that this equation can be cast into Hirota's bilinear form as

$$D_x D_t (f,f) = H(f) \; ,$$

where H is a function that you should find. Verify that

$$f = 1 + \exp(ax + bt + c)$$

is a solution, provided that the constants a, b, c satisfy appropriate relations that you should find. What is the velocity of this solution?

Q7 Consider $u\,(x,t)$ that satisfies the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

(a) If $D = d/dx$ and $f\,(x)$ is a general function of x, show that

$$[D, f] = f_x,$$
$$\left[D^2, f\right] = f_{xx} + 2f_x D.$$

(b) Consider operators

$$L = D^2 + u\,(x,t)\,,$$
$$B = -4D^3 + \alpha_1\,(x)\,D + \alpha_0\,(x)\,,$$

Find the functions $\alpha_1\,(x)$ and $\alpha_0\,(x)$ of u and u_x such that the KdV equation can be written in the form

$$L_t + [L, B] = 0.$$

You may use that: $[D^3, f] = f_{xxx} + 3f_{xx}D + 3f_x D^2$ *for any function* $f = f\,(x)$*, together with the formulas in part 7(a).*

Q8 Consider the Schrödinger equation

$$\psi'' + \left(k^2 - V(x)\right)\psi = 0,$$

with potential

$$V(x) = \lambda_1 \delta(x + a) - \lambda_2 \delta(x - a),$$

where $\lambda_1, \lambda_2 > 0$.

(a) By imposing the appropriate matching conditions, derive the equation for bound state solutions $k = i\mu$ with $\mu > 0$ in the form

$$(b + 2\mu)(c - 2\mu) - \lambda_1 \lambda_2 e^{\mu d} = 0,$$

giving the constants b, c and d.

(b) Find the bound state solutions (if any) as:

(i) $a \to 0$

(ii) $a \to \infty$

In each case give the corresponding constraints (if any) on λ_1 and λ_2.

(c) Taking for simplicity $\lambda_1 = \lambda_2$ and $a > 0$, is it possible to conclude that there always exists a bound state?