EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3251-WE01

Title:

Stochastic Processes III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $X : \Omega \to \mathbb{R}$ an integrable random variable, i.e. $\mathsf{E}[|X|] < \infty$, and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$.

(a) State the definition of the abstract conditional expectation $\mathsf{E}[X|\mathcal{G}]$.

(b) Using (a), show that if $\mathcal{G} = \{\emptyset, \Omega\}$ is the trivial σ-algebra, then $\mathsf{E}[X|\mathcal{G}] = \mathsf{E}[X]$ almost surely.

(c) Suppose $\mathsf{E}[X|\mathcal{G}] = \mathsf{E}[X]$ almost surely. Does this imply $\mathcal{G}$ is the trivial σ-algebra? Give a proof if this is the case, or a counterexample otherwise.

Q2 This question deals with Poisson processes.

(a) Customers arrive at a store according to a Poisson process of rate 10/hour. Each customer is independently a little spender with probability 3/4 or a big spender with probability 1/4. A little spender spends on average 5 pounds and a big spender spends on average 20 pounds. Let T be the total amount of money earned by the shop in the first 5 hours. Find $\mathsf{E}[T]$.

(b) Consider two independent Poisson processes, one consisting of red balls and the other of blue balls, both having rate λ. Find the probability that 3 red balls appear before 3 blue balls appear.

Q3 This question deals with martingales.

(a) Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable such that $\mathsf{E}[|X|] < \infty$. Let $\mathcal{F}_n$ for $n \geq 0$ be a filtration, that is, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$ is a nested sequence of σ-algebras. Define $M_n = \mathsf{E}[X \mid \mathcal{F}_n]$. Show that M_n is a martingale with respect to the filtration $\mathcal{F}_n$. Make sure to verify all three martingale conditions.

(b) Suppose that M_n is a martingale with respect to the filtration $\mathcal{F}_n$. Suppose also that $\mathsf{E}[M_n^2] < \infty$ for every n. Show that for $i < j$, $\mathsf{E}[(M_j - M_i)^2] = \mathsf{E}[M_j^2] - \mathsf{E}[M_i^2]$.

Q4 Let $(Z_n)_{n\geq 0}$ be a branching process with $Z_0 = 1$, with offspring distribution having mean $m = 1$ and finite variance $\sigma^2 < \infty$. For each of the statements below, provide a proof if it is correct, or give a counterexample otherwise.

(a) $\mathrm{Var}(Z_n) = \sigma^2 n$ for all $n \in \mathbb{N}$.

(b) $\lim_{n\to\infty} \mathsf{P}(Z_n = 0) = 1$.

SECTION B

Q5 Denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of all non-negative integers. Let U, V be two $\mathbb{N}_0$-valued random variables with probability mass functions

$$\mathsf{P}(U = n) = p(1-p)^n \quad \text{and} \quad \mathsf{P}(V = n) = e^{-\lambda}\frac{\lambda^n}{n!} \qquad \forall n \in \mathbb{N}_0$$

where $p \in (0,1)$ and $\lambda > 0$.

5.1 Show that if U stochastically dominates V, then $e^{-\lambda} \geq p$.

5.2 By considering suitable events of a Poisson process or using direct computation, explain why

$$\mathsf{P}(V \geq n) = \mathsf{P}(Z_1 + \cdots + Z_n \leq \lambda) \qquad \forall n \in \mathbb{N}$$

where $Z_i \overset{i.i.d.}{\sim} \text{Exp}(1)$. Hence, or otherwise, show that if $e^{-\lambda} \geq p$, then U stochastically dominates V.

5.3 Let W be another random variable with generating function

$$\mathsf{E}[s^W] = \left[\frac{p}{1-(1-p)s}\right]^{2023} \qquad \forall s \in (0,1)$$

with $e^{-\lambda} \geq p^{2023}$. Show that W stochastically dominates V. (Hint: what is the generating function for U?)

Q6 Consider a renewal process $M(t) := \sum_{n\geq 0} 1_{\{S_n \leq t\}}$ where $(S_n)_{n\geq 0}$ is a random walk with delay distribution $\lambda(t) := \mathsf{P}(S_0 \leq t)$ satisfying $\mathsf{P}(S_0 \geq 0) = 1$, and increment distribution $S_{n+1} - S_n \overset{i.i.d.}{\sim} \text{Uniform}([0,1])$. Let $m(t) := \mathsf{E}[M(t)]$ be the renewal function.

6.1 Derive the renewal equation satisfied by $m(t)$.

6.2 Show that in the zero delay case (i.e. $\mathsf{P}(S_0 = 0) = 1$),

$$m'(t) = \begin{cases} m(t) & \text{for } t \in (0,1), \\ m(t) - m(t-1) & \text{for } t > 1, \end{cases}$$

and hence

$$m(t) = \begin{cases} e^t & \text{for } t \in [0,1], \\ e^t - e^{t-1}(t-1) & \text{for } t \in [1,2]. \end{cases}$$

6.3 It is known that for a certain delay distribution λ, the renewal function $m(t)$ is proportional to t for all $t \geq 0$. Find a formula for $\lambda(t)$.

Please quote any results used and justify all steps carefully in your answer.

Q7 Let X_t be a continuous time Markov process on the state space $\mathcal{I} = \{1, 2, 3\}$ with generator (Q-matrix)

$$Q = \begin{pmatrix} -12 & 12 & 0 \\ 4 & -10 & 6 \\ 0 & 8 & -8 \end{pmatrix}$$

7.1 Show that this Markov process is irreducible.

7.2 Find the characteristic polynomial of Q and identify the eigenvalues.

7.3 Find the invariant distribution π of the process.

7.4 Compute $p_{2,3}(t)$ exactly.

Q8 Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution

$$\mathsf{P}(X_i = +1) = \mathsf{P}(X_i = -1) = 1/2.$$

Let $S_n = \sum_{k=1}^{n} X_k$ for $n \geq 1$ and $S_0 = 0$. Consider the σ-algebras $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \geq 1$ and let $\mathcal{F}_0$ be the trivial σ-algebra.

8.1 Let $M_n = S_n^4 - 6nS_n^2 + 3n^2 + 2n$ for $n \geq 0$. Show that M_n is a martingale with respect to the filtration $\mathcal{F}_n$. Carefully verify all three martingale conditions.

8.2 State the definition of a stopping time T with respect to the filtration $\mathcal{F}_n$. State any version of the Optional Stopping Theorem. For a positive integer K, define $T = \inf\{n \geq 0 : |S_n| = K\}$. Show that T is a stopping time.

8.3 For a positive integer K, let $T = \inf\{n \geq 0 : |S_n| = K\}$. You may use the fact from lectures that $\mathsf{E}[T] = K^2$. Find $\mathsf{E}[T^2]$. Carefully justify all steps in your calculation by quoting appropriate theorems.