EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3281-WE01

Title:
Topology III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Let $X = \{1, 2, 3, 4, 5\}$, and consider the collections of subsets

$$\mathcal{A} = \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}\} \text{ and } \mathcal{B} = \{\{1, 2, 3, 4\}, \{3\}, \{4\}, \{5\}\}.$$

(a) Find $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, the smallest topologies containing each collection. Is $\mathcal{A}$ a basis for $\tau_{\mathcal{A}}$? Is $\mathcal{B}$ a basis for $\tau_{\mathcal{B}}$? Explain briefly.

(b) For each topology, determine:
whether it is compact or not;
whether it is Hausdorff or not;
whether it is connected or not.

Q2 Let d be any metric on a non-empty set X. Consider the functions:

$$d_{\min} : X \times X \to \mathbb{R} \quad \text{given by } d_{\min}(x, y) = \min\{1, d(x, y)\}$$

$$d_{\max} : X \times X \to \mathbb{R} \quad \text{given by } d_{\max}(x, y) = \max\{1, d(x, y)\}$$

(a) Show that the triangle inequality holds for both these functions, but that only one of them is a metric.

(b) For $x \in X$ and $r \in [0, \infty)$, the open ball $B(x; r)$ is defined by

$$B(x; r) = \{y \in X \mid d(x, y) < r\}.$$

We can define corresponding open balls $B_{\min}(x; r)$, $B_{\max}(x; r)$, for $d_{\min}$, $d_{\max}$ respectively. In terms of the $B(x; r)$, describe the $B_{\min}(x; r)$ and $B_{\max}(x; r)$ for different values of r.

Q3 (a) State what it means for two topological spaces to be homotopy equivalent.

(b) Let X, Y and Z be topological spaces and suppose that X and Y are homotopy equivalent. Show that $X \times Z$ is homotopy equivalent to $Y \times Z$.

Q4 (a) In terms of a suitable graph, state the definition of the Euler characteristic of a closed surface. Briefly justify why the definition is independent of your choice of (suitable) graph.

(b) If closed surfaces S_1 and S_2 are homeomorphic, prove that their Euler characteristics agree; that is, $\chi(S_1) = \chi(S_2)$.

SECTION B

Q5 (a) Prove that a compact subset A of a Hausdorff space X is closed in X.

(b) State the Heine-Borel theorem about compactness in $\mathbb{R}^n$ with the standard topology. Prove **one** direction of this result.

(c) Let

$$S = \{f(t) \mid t \in [1, \infty)\} \subseteq \mathbb{R}^2 \text{ where } f(t) = \left(1 + \frac{1}{t}\right)(\cos(2\pi t), \sin(2\pi t)).$$

Which points in $\mathbb{R}^2 \setminus S$ are limit points of S? Show that these are limit points; you are not asked to show that these are the only ones.

(d) Show in **two** ways that S is not compact: i) using Heine-Borel;
ii) directly, using the definition of compactness. [You may use visual or geometric arguments, but for full marks define any sets explicitly.]

Q6 Let

$$C = \{(\cos\theta, \sin\theta, t) \mid \theta \in [-\pi, \pi], t \in [-1, +1]\} \subseteq \mathbb{R}^3 \text{ (with standard topology)},$$

and define the following two equivalence relations on C:

$$(x_1, y_1, z_1) \sim_1 (x_2, y_2, z_2) \iff \begin{cases} (x_1, y_1, z_1) = (x_2, y_2, z_2) \text{ OR} \\ x_1 = x_2, y_1 = y_2, \{z_1, z_2\} = \{-1, +1\} \end{cases}$$

$$(x_1, y_1, z_1) \sim_2 (x_2, y_2, z_2) \iff \begin{cases} (x_1, y_1, z_1) = (x_2, y_2, z_2) \text{ OR} \\ z_1 = z_2 = -1 \text{ OR } z_1 = z_2 = +1 \end{cases}$$

(a) The quotient spaces $C/{\sim_1}$ and $C/{\sim_2}$ are homeomorphic to two well-known surfaces S_1 and S_2. *Without proof* say which these are.

(b) Consider a function $f : C \to \mathbb{R}^3$ of the form

$$f(\cos\theta, \sin\theta, t) = (k(t)\cos\theta, k(t)\sin\theta, t).$$

For a suitable function $k : [-1, +1] \to \mathbb{R}$, the image of f is homeomorphic to one of the surfaces S_1 or S_2 from part (a). Decide which, and find such a function k.

From now on we write this surface and the correponding equivalence as S and $\sim$.

(c) For certain subsets A and B of C, the space $C/{\sim}$ could also be defined as $(C/A)/B$. Find these subsets. Why could we not write $C/{\sim}$ as $C/(A \cup B)$? Describe $C/(A \cup B)$ in words, or a picture.

(d) As usual, we define the maps

$$\pi : C \to C/{\sim} \text{ given by } \pi(x, y, z) = [(x, y, z)],$$

$$\overline{f} : C/{\sim} \to S \text{ given by } \overline{f}[(x, y, z)] = f(x, y, z),$$

so that $f = \overline{f} \circ \pi$. Show that $\overline{f}$ is both well-defined and injective.

(e) Show that $\overline{f} : C/{\sim} \to S$ is a homeomorphism.
You may use the following result from lectures: If X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.
If you use any other results then refer to them clearly.

Q7 (a) State the definitions of (i) a finite simplicial complex and (ii) a triangulation of a topological space.

(b) Let K and L be 2-dimensional, finite simplicial complexes and suppose that $f : K \to L$ is a simplicial map such that, for some fixed $r \in \mathbb{N}$, every simplex in L is the homeomorphic image of exactly r disjoint simplices in K. We say that K is an r-fold covering of L. Find a relationship between r and the Euler characteristics of K and L.

(c) Suppose that the sphere S^2 is triangulated by a finite simplicial complex K, where K is an r-fold covering of a simplicial complex L. Determine all possible values of $r \in \mathbb{N}$ and, if L also triangulates a closed surface M, determine all possibilities for M up to homeomorphism.

Q8 Let X be a topological space and let $x_0 \in X$.

(a) State the definition of the fundamental group $\pi_1(X, x_0)$ of the based space (X, x_0), including the definition of the group operation.

(b) If X is a path-connected space with non-trivial fundamental group $\pi_1(X, x_0)$, show that the identity map $\mathrm{id}_X : X \to X$ is not null homotopic.

(c) Suppose now that the topology on X is induced by a metric $d_X : X \times X \to [0, \infty)$ and let ΩX be the set whose elements are continuous maps $f : [0, 1] \to X$ with $f(0) = f(1) = x_0$, which is equipped with a metric

$$\begin{aligned} D : \Omega X \times \Omega X &\to [0, \infty) \\ (f, g) &\mapsto \sup\{d_X(f(t), g(t)) \mid t \in [0, 1]\}\,. \end{aligned}$$

If the fundamental group $\pi_1(X, x_0)$ is trivial, prove that ΩX is path connected with respect to the topology induced by the metric D, being careful to ensure that any maps you introduce are continuous.