Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3291-WE01

Title: Partial Differential Equations III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Let us consider the following Cauchy problem associated to a first order PDE

$$\begin{cases} x_2 \partial_{x_1} u(x_1, x_2) = 1, & (x_1, x_2) \in \mathbb{R}^2, \\ u(0, x_2) = 0, & x_2 \in \mathbb{R}. \end{cases} \tag{1}$$

1.1 Identify the leading vector field, the Cauchy data and the Cauchy curve.

1.2 Are the points on the Cauchy curve characteristic or non-characteristic? Justify your answer.

1.3 Using the method of characteristics, solve the problem in (1). Give the domain of definition of the solution.

Q2 Consider the Cauchy problem for Burgers' equation

$$\begin{cases} \partial_t u(x,t) + \frac{1}{2}\partial_x(u^2(x,t)) = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is given.

2.1 Let $u_0(x) = \frac{1}{7}x^7$. Show that (2) has a global in time classical solution.

2.2 Let $u_0(x) = \sin(x)$. Write down the definition of the critical time t_c (until when we can guarantee the existence of a classical solution to (2)) associated to this initial datum. Show that $t_c \leq 1$.

Q3 In this problem we consider harmonic function on the unit ball in $\mathbb{R}^3$, $B_1(0)$.

3.1 Using the fact that the Laplacian in spherical coordinates is given by

$$\Delta \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

show that if $u \in C^2\left(\overline{B_1(0)}\right)$ is radial (i.e. only depends on r in spherical coordinates) and harmonic in $B_1(0)$ then it must be constant.

3.2 Show that there exists no radial solution in $C^2\left(\overline{B_1(0)}\right)$ to the equation

$$\begin{cases} -\Delta u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in B_1(0), \\ u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \partial B_1(0), \end{cases}$$

for $f(\boldsymbol{x}) = x_1^2$.

Q4 Consider the heat-like equation

$$\begin{cases} u_t - u_{xx} + cu = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases} \tag{3}$$

where $c \in \mathbb{R}$ is a fixed constant and $g \in C_c(\mathbb{R})$.

4.1 Define $v(x,t) = e^{ct}u(x,t)$. Show that $v(x,t)$ solves the heat equation

$$\begin{cases} v_t - v_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ v(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

4.2 Show that there exists a solution to (3) that satisfies

$$\sup_{x\in\mathbb{R}} |u(x,t)| \leq e^{-ct} \|g\|_{L^\infty(\mathbb{R})}.$$

You may use the following inequality without proof: For any $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ we have that

$$\left| \int_{\mathbb{R}} f(x-y)\, g(y) \right| \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})}, \qquad \forall x \in \mathbb{R}.$$

SECTION B

Q5 We consider the following conservation law

$$\begin{cases} \partial_t u(x,t) - u(x,t)\partial_x u(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{4}$$

[Notice that this is *not* Burgers' equation.]

5.1 Suppose that u_0 is bounded, differentiable with bounded derivative. Give a formula of the critical time t_c, for which we know that (4) has a classical solution on $\mathbb{R} \times (0, t_c)$.

5.2 Let $u_0(x) = -\arctan(x)$. Show that in this case (4) has a global in time classical solution.

5.3 Let u_0 now be given by

$$u_0(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

By drawing the characteristics, show that there is instantaneous crossing of characteristics. Find a shock that satisfies the Rankine–Hugoniot condition. Give the expression of the weak solution in this case.

Q6 We aim to solve the following problem by the method of characteristics

$$\begin{cases} \partial^2_{xx} u - 3\partial^2_{xy} u + 2\partial^2_{yy} u = 0, & (x,y) \in \mathbb{R}^2, \\ u(1,y) = g(y), & y \in \mathbb{R}, \\ \partial_x u(1,y) = h(y), & y \in \mathbb{R}, \end{cases} \tag{5}$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are given smooth functions.

6.1 Identify the Cauchy data and the Cauchy curve in the above problem.

6.2 Rewrite the PDE in (5) as a system of two linear first order PDEs. [*Hint*: think about the algebraic relation $(a-b)(a-2b) = a^2 - 3ab + 2b^2$, $(a, b \in \mathbb{R})$.]

6.3 By solving the two first order PDEs arising from **6.2** using the method of characteristics, find the solution to (5).

Q7 Let Ω be an open bounded set with smooth boundary in $\mathbb{R}^n$ and let u_1 and u_2 be $C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ solutions to Poisson equation

$$\begin{cases} -\Delta u_i(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u_i(\boldsymbol{x}) = g_i(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega, \end{cases}$$

$i = 1, 2$, where $f \in C^1\left(\overline{\Omega}\right)$ and $g_1, g_2 \in C(\partial\Omega)$.

7.1 Show that for any $\boldsymbol{x} \in \overline{\Omega}$

$$u_2(\boldsymbol{x}) - u_1(\boldsymbol{x}) \leq \max_{\boldsymbol{x} \in \partial\Omega} \left(g_2(\boldsymbol{x}) - g_1(\boldsymbol{x})\right).$$

7.2 Show that

$$\max_{\boldsymbol{x} \in \overline{\Omega}} |u_2(\boldsymbol{x}) - u_1(\boldsymbol{x})| \leq \max_{\boldsymbol{x} \in \partial\Omega} |g_2(\boldsymbol{x}) - g_1(\boldsymbol{x})|.$$

7.3 For $n \in \mathbb{N}$ let $u_n \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ solve the system

$$\begin{cases} -\Delta u_n(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u_n(\boldsymbol{x}) = g_n(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega, \end{cases}$$

and let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ solve the system

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega. \end{cases}$$

Show that if $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly to g on $\partial\Omega$ then $\{u_n\}_{n\in\mathbb{N}}$ converges uniformly to u on $\overline{\Omega}$.
Recall that we say that a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $C(K)$ converges uniformly to $f \in C(K)$ if

$$\sup_{x\in K} |f_n(x) - f(x)| \underset{n\to\infty}{\longrightarrow} 0.$$

Q8 Let $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a classical solution to the equation

$$\begin{cases} u_t + k u_{xxxx} = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where $k > 0$ is a fixed constant and f is a smooth function on $\mathbb{R}$ that belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

8.1 Show that $\widehat{u}$, the Fourier transform of u in the $x-$variable, satisfies

$$\widehat{u}(\xi, t) = \widehat{f}(\xi) e^{-k\xi^4 t}.$$

8.2 Using the fact that the Fourier transform preserves the L^2 norm (Plancherel's identity) show that

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq \frac{\int_{\mathbb{R}} e^{-x^4} dx}{\sqrt[4]{2kt}} \|f\|^2_{L^2(\mathbb{R})}.$$