Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
|---|---|---|
| May/June | 2023 | MATH3301-WE01 |

| Title: |
|---|
| Mathematical Finance III |

| | | |
|---|---|---|
| Time: | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | Yes | Models Permitted: Casio FX83 series or FX85 series. |

| | |
|---|---|
| Instructions to Candidates: | Answer all questions.<br>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.<br>Students must use the mathematics specific answer book. |

| | |
|---|---|
| Revision: | |

ED01/2023

## SECTION A

**Q1** Consider the market consisting of one risk-free asset with price dynamics $B_t = (6/5)^t$, $t = 0, 1, 2$, and one risky asset whose price $S_t$ evolves with $S_0 = 15$, $u = 1.4$ and $d = 0.6$.

(a) Does this market contain an opportunity for arbitrage?

(b) A broker offers a European put option on the risky asset with expiry date $T = 2$ and strike price $K = 15$. Construct a hedging portfolio for the option.

(c) The option can be bought for £1. Is this a fair price? Either justify that it is, or describe how to construct a portfolio involving the option which returns a guaranteed profit.

**Q2** (a) State the Cox-Ross-Rubinstein formula for the fair price at time $T - t$ of a European call option with expiry date $T$ and strike price $K$.

(b) Consider a market in which there is a stock whose initial share price is $S_0 = 48$, and at each of times $t = 1, 2, 3$, the share price either increases by 12 or decreases by 12 (so that, for example, at $t = 2$, the possible share prices are 72, 48, and 24). Suppose that the risk-free interest rate for this market is $r = 0.1$.

We are interested in the fair price of a European call option with strike price $K = 48$ and expiry time $T = 3$.

(i) Can we use the Cox-Ross-Rubinstein formula to calculate the price? Justify your answer.

(ii) What is the fair price of this option at time 1, if $S_1 = 60$?

**Q3** (a) State the definition of *Brownian motion*;

(b) Let $(W_t)_{t\geq 0}$ be a Brownian motion and $a > 0$. Prove that

$$\left(\frac{1}{\sqrt{a}} W_{at}\right)_{t\geq 0} \text{ is a Brownian motion;}$$

(c) State the definition for a stochastic process $(X_t)_{t\geq 0}$ to be *a martingale with respect to a given filtration* $(\mathcal{F}_t)_{t\geq 0}$;

(d) Let $(W_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by $(W_t)_{t\geq 0}$. Prove that

$$(W_t)_{t\geq 0} \text{ is a martingale with respect to } (\mathcal{F}_t)_{t\geq 0}.$$

**Q4** Let $(W_t)_{t\geq 0}$ be a Brownian motion.

(a) State the rules of Itô calculus (box calculus) among $\mathrm{d}t$ and $\mathrm{d}W_t$, and state also the Itô formula for $\mathrm{d}F(t, S_t)$ with a smooth function $F(t, x)$ and Itô process $(S_t)_{t\geq 0}$;

(b) Apply the Itô formula to $\mathrm{d}(W_t^2)$ and express $\mathrm{d}(W_t^2)$ in terms of $\mathrm{d}t$, $\mathrm{d}W_t$ and $W_t$;

(c) Prove

$$\int_0^t W_s \mathrm{d}W_s = \frac{1}{2}(W_t^2 - t)\ .$$

## SECTION B

**Q5** (a) Give the definitions of American call and put options.

(b) Given a (general) European claim, and the corresponding American claim, is one worth more than the other? Why?

(c) Consider the binomial market model with 3 periods, in which $S_0 = 100$, and the share price evolves with $u = 1.2$ and $d = 0.8$. Suppose that the risk-free interest rate is $r = 0.1$. What is the fair price at time 0 for an American put option with strike price $K = 100$? Under which circumstances is it wise to exercise the option early?

**Q6** (a) What does it mean for a random variable $X : \Omega \to \mathbb{R}$ to be *measurable* with respect to a $\sigma$-algebra $\mathcal{F}$?

(b) A random variable $\tau$ is a *stopping time* with respect to a discrete-time filtration $\mathcal{F}_n$ if, for every $n$, the indicator random variable $\mathbb{1}\{\tau \leq n\}$ is measurable with respect to $\mathcal{F}_n$. In the context of the multi-period binomial model, give an example of:

(i) a random variable that *is* a stopping time.

(ii) a random variable that *is not* a stopping time.

(c) Show that the fair price for an American put option, $P^A$, with expiry date $T$, strike price $K$ and interest rate $r$, satisfies

$$K(1 + r)^{-T} - S_0 \leq P^A \leq K(1 + r)^{-T},$$

whatever happens to the share price.

**Q7** Let $B_t = e^{rt}$ be the bond price with interest rate $r > 0$ and $(S_t)_{t\geq 0}$ be the stock price following the Black-Scholes-Merton SDE:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad \mu \in \mathbb{R}, \quad \sigma > 0, \quad S_0 = x_0 > 0, \quad t \in [0, T] \ .$$

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by $(W_t)_{t\in[0,T]}$.

(a) Prove $\mathrm{d}S_t \mathrm{d}S_t = \sigma^2 S_t^2 \mathrm{d}t$;

(b) Let $(a_t, b_t)$ be a portfolio and $V_t = a_t B_t + b_t S_t$ be the value process. State the definition of *the self-financing condition*;

(c) Let $\Phi = \Phi(S_T)$ be a contingent claim with expiry time $T$. State the definition of the portfolio $(a_t, b_t)$ *replicating* $\Phi$;

(d) Let $\Pi_t(\Phi)$ be the fair price (no-arbitrage price) of the contingent claim $\Phi$ at $t$, and let $F(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a smooth function so that $F(t, S_t) = \Pi_t(\Phi)$. Prove that $F(t, x)$ is given by the following partial differential equation:

$$\begin{cases} \partial_t F + \frac{1}{2}\sigma^2 x^2 \partial_{xx} F + rx\partial_x F - rF = 0 \\ F(T, x) = \Phi(x) \ . \end{cases}$$

**Q8** Let $B_t = e^{rt}$ be the bond price with interest rate $r > 0$ and $(S_t)_{t\geq 0}$ be the stock price following the Black-Scholes-Merton SDE:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad \mu \in \mathbb{R}, \quad \sigma > 0, \quad S_0 = x_0 > 0, \quad t \in [0, T] \ .$$

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by $(W_t)_{t\in[0,T]}$. Let $(\tilde{W}_t)_{t\geq 0}$ be the Brownian motion under the risk-neutral measure (martingale measure) $\mathbb{Q}$.

(a) Write down the formula expressing $(\tilde{W}_t)_{t\geq 0}$ in terms of $(W_t)_{t\geq 0}$, $\mu$, $\sigma$, $r$, $t$;

(b) Prove that the process $\tilde{S}_t := e^{-rt} S_t$ is martingale with respect to $\mathbb{Q}$;

(c) State Martingale Representation Theorem;

(d) Let $\Phi$ be a contingent claim with expiry t time $T$. Let $\Pi_t(\Phi)$ be the fair price (no-arbitrage price) of the contingent claim $\Phi$ at $t$. Prove

$$\Pi_t(\Phi) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[\Phi | \mathcal{F}_t].$$