Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3391-WE01

Title: Quantum Computing III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

University of Durham Copyright

SECTION A

Q1 Given the single qubit Hilbert space, consider the two unitary time evolutions

$$\hat{U}_1 = \sigma_1\sigma_2\sigma_3\,,$$
$$\hat{U}_2 = \sigma_3\sigma_2\sigma_1\,,$$

where the Pauli matrices σ_j are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,.$$

(a) How does the pure qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ transform under the time evolution $\hat{U}_1$?

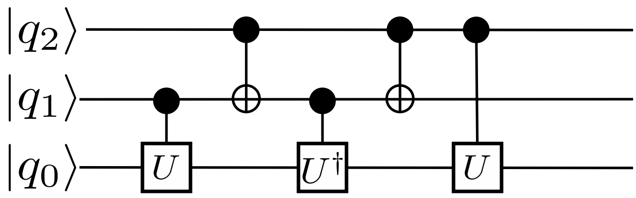
(b) Similarly, how does the same state $|\psi\rangle$ transform under the time evolution $\hat{U}_2$?

Q2 Discuss which of the following 2×2 matrices can describe a possible qubit density matrix, and whether the corresponding qubit state is pure or mixed. Justify your answer.

$$\rho_1 = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\,, \qquad \rho_2 = \frac{1}{4}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\,, \qquad \rho_3 = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\,.$$

Q3 Show that the gate set $\{\mathrm{NOR}, \mathrm{CNOT}\}$ (where $\mathrm{NOR}(x, y) := \mathrm{NOT}(x\,\mathrm{OR}\,y)$) is a universal gate set for classical computation. You can assume that $\{\mathrm{NOT}, \mathrm{AND}, \mathrm{OR}, \mathrm{CNOT}\}$ is a universal gate set. Explain why $\{\mathrm{NOR}, \mathrm{CNOT}\}$ cannot be a universal gate set for quantum computation.

Q4 **4.1** Show that the following circuit, written in terms of CNOT, controlled-U and controlled-$U^\dagger$ gates is equivalent to a CC-U^2 gate (controlled-controlled-U^2, note that it is U^2 and not U here), with q_1 and q_2 the control qubits and q_0 the target, for any unitary U:



4.2 Write the matrix representation of the action of this circuit in the computational basis.

4.3 Find a unitary U such that the CC-U^2 circuit above reduces to the quantum Toffoli (CCNOT) gate.

University of Durham Copyright

SECTION B

Q5 Consider the 2-qubit operator

$$\hat{U} = \frac{1}{2}\Big(\hat{I} \otimes \hat{I} + \sum_{j=1}^{3} \sigma_j \otimes \sigma_j\Big)\,,$$

where $\hat{I}$ denotes the single qubit identity operator and the Pauli matrices σ_j are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,.$$

5.1 Show that $\hat{U}$ is unitary and evaluate the action of $\hat{U}$ on the computational basis elements $|x\rangle \otimes |y\rangle$, with $x, y \in \{0, 1\}$.

5.2 Consider now the 3-qubit state

$$|0\rangle \otimes |\beta_{00}\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}\Big(\,|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\,\Big)\,,$$

in the bipartite system where Alice has access only to the first two qubits, while Bob can only act on the third qubit. What are the possible outcomes, with their respective probabilities and final states, if firstly Alice evolves her two qubits with the unitary operator $\hat{U}$ defined above, and then she measures σ_1 on her first qubit and σ_3 on her second qubit?

Q6 Consider the ensemble $\{(p_0, |0\rangle), (p_1, |1\rangle), (p_+, |+\rangle)\}$, i.e. the mixture of $|0\rangle$ with probability $0 \leq p_0 \leq 1$, $|1\rangle$ with probability $0 \leq p_1 \leq 1$, and $|+\rangle$ with probability $0 \leq p_+ \leq 1$, subject to $p_0 + p_1 + p_+ = 1$.

[Remember $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$.]

6.1 Write the corresponding density operator, $\hat{\rho}$, and its 2×2 matrix representation, ρ, using the mapping of the computational basis ket vectors to the standard basis vectors $|0\rangle \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

6.2 For which values of p_0, p_1, p_+, if any, does $\hat{\rho}$ describe a pure state?

6.3 Consider again the ensemble $\{(p_0, |0\rangle), (p_1, |1\rangle), (p_+, |+\rangle)\}$ for which we fixed now $p_0 = p_1 = p$ and $p_+ = 1 - 2p$, with $0 \leq p \leq \frac{1}{2}$. What is the expectation value of the observable σ_3 on this state?

University of Durham Copyright

Q7 Consider a two-qubit system. We wish to construct a circuit to realize the operation

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

7.1 Decompose this operator as

$$U = U_{1,2}U_{2,3}$$

where $U_{i,j}$ are unitary operators which only act non-trivially in the subspace of the Hilbert space spanned by the computational basis vectors $|i\rangle$ and $|j\rangle$. Our conventions are that the columns are labelled (reading from left to right) 0, 1, 2 and 3, and similarly for the rows (reading from top to bottom).

7.2 Write the operators $U_{i,j}$ above in terms of controlled-unitary operators. [*Hint: you might want to use the Gray code* $01 \to 11 \to 10$.]

7.3 Write all the controlled-unitary gates you found above in terms of CNOT gates and the single qubit Hadamard gate $H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

[*Hint: you might find the equality* $XHXH = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *useful, where* $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.]

7.4 Draw the resulting quantum circuit.

Q8 We want to encode a single logical qubit in four physical qubits so as to protect against arbitrary single bit flip errors.

8.1 Show that $|\bar{0}\rangle = |0000\rangle$, $|\bar{1}\rangle = |1111\rangle$ provides a suitable encoding, by showing that the single bit flip errors map the code subspace to orthogonal subspaces.

8.2 Find a set of three error syndromes that distinguish the code subspace and the subspaces arising from single bit flip errors.

8.3 Show that $\bar{X} = X_0X_1X_2X_3$ acts as the NOT operation on the logical qubit.

University of Durham Copyright