Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3471-WE01

Title: Geometry of Mathematical Physics III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

SECTION A

Q1 (a) State the definition of a representation of a Lie group. When is a representation called irreducible?

(b) For the groups $U(1)$ and $SO(3)$, describe two different examples of irreducible representations each.

(c) Let $g \in SO(3)$ act on $\mathbb{R}^3$ by letting

$$\boldsymbol{v} \to r(g)\boldsymbol{v} = g^2\boldsymbol{v}$$

for $\boldsymbol{v} \in \mathbb{R}^3$. Does this define a representation? Explain your reasoning.

Q2 Consider the path

$$g(t) = \exp\left(it\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right)$$

for $t \in \mathbb{R}$ in $SU(2)$.

(a) Find the tangent vector at the identity element of $SU(2)$ associated with $g(t)$.

(b) Find the matrix $g(t)$ by working out the exponential. Check that the result is in $SU(2)$.
hint:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

(c) Show that the set $\{g(t)|t \in \mathbb{R}\}$ describes a subgroup of $SU(2)$. Which group is this subgroup isomorphic to?

Q3 Consider a relativistic field theory with two complex scalar fields ϕ^1 and ϕ^2 of electric charge 1 and a complex scalar field S of electric charge -2, coupled to a $U(1)$ gauge field A_μ.

(a) Write a gauge invariant kinetic Lagrangian density for the above fields. What is the global symmetry of this field theory?

(b) Write down the most general gauge invariant scalar potential that respects the global symmetry found in part (a) and that is a polynomial of degree at most three in the fields and their complex conjugates.

Q4 Let A_μ be an $SO(3)$ gauge field, $F_{\mu\nu}$ be its field strength, and Y be a real 3-vector that transforms as $Y \mapsto MY$ under an $SO(3)$ gauge transformation with group element $M \in SO(3)$. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}) - \frac{1}{2}(D^\mu Y)^T(D_\mu Y) - (Y^T Y)^2 \ ,$$

where D_μ is the gauge covariant derivative defined by $D_\mu Y = \partial_\mu Y - iA_\mu Y$, and the subscript T denotes the matrix transpose.

Write the finite $SO(3)$ gauge transformations of A_μ and $F_{\mu\nu}$ in terms of M and M^T (you may use the expression for $F_{\mu\nu}$ in terms of covariant derivatives). Show that the Lagrangian density $\mathcal{L}$ is invariant under such gauge transformations.

SECTION B

Q5 The Lorentz group L in three dimensions is the group of matrices such that $\Lambda^T\eta\Lambda = \eta$. Here η is a diagonal matrix with non-zero components $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = 1$.

5.1 Find a basis of the Lie algebra of L and work out the commutators between the basis elements.

5.2 Let s_i for $i = 0, 1, 2$ be matrices such that

$$\{s_i, s_j\} = 2\eta_{ij}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Find matrices s_i that satisfy this relation.

hint: try matrices proportional to the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

5.3 Let $\mathfrak{s}$ be the Lie algebra with basis vectors $S_{ij} := [s_i, s_j]$, and with $[S_{ij}, S_{kl}] \equiv S_{ij}S_{kl} - S_{kl}S_{ij}$ the usual commutator between matrices. Find a Lie algebra homomorphism from $\mathfrak{s}$ to the Lie algebra of L.

5.4 Does the Lie algebra representation constructed above arise from a group representation of L? Explain your answer.

Q6 Let

$$(\boldsymbol{x}, \boldsymbol{y}) := \sum_{i=1}^{2n} \sum_{j=1}^{2n} x_i \Omega_{ij} y_j$$

be a bilinear form on $\mathbb{C}^{2n}$ with $\Omega_{ij} \in \mathbb{C}$. Let $G \subset GL(2n, \mathbb{C})$ be the set of matrices M for which $(M\boldsymbol{x}, M\boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^{2n}$.

6.1 Show that G is a Lie group using matrix multiplication as the group operation.

6.2 Under which condition is a matrix contained in the Lie algebra $\mathfrak{g}$ of G? Verify that matrices obeying this condition form a vector space.

6.3 For the case $n = 1$ and $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ show that $G \cap U(2) = SU(2)$.

6.4 For the case $n = 2$ and $\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ find the dimension of G.

Q7 An abelian gauge field A_μ in one time and two space dimensions has Lagrangian density

$$\mathcal{L} = k\ \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu\ ,$$

where k is a real constant, $\epsilon^{\mu\nu\rho}$ is the totally antisymmetric tensor with three indices, normalized such that $\epsilon^{012} = 1$, and J^μ is a conserved current.

7.1 Show that the action $S[A_\mu] = \int d^3x\ \mathcal{L}$ changes by a boundary term under a gauge transformation $A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \alpha(x)$. (You can leave the boundary term as an integral over spacetime.)

7.2 Find the equations of motion and show that they imply $\partial_\mu J^\mu = 0$.

7.3 Express the equations of motion in terms of the electric field $\boldsymbol{E}$, with components $E_i = F_{i0}$, and the magnetic field $B = F_{12}$. Find the total charge Q associated to the current J^μ for a field configuration with magnetic flux $\Phi = \int d^2x\ B$ across space.

Q8 It is given that, in a gauge where $A_0 = 0$, the energy of a static field configuration of a gauge field A_μ and a scalar ϕ transforming in the adjoint representation of the gauge group is

$$E = \int d^3x \ \mathrm{tr}\left(E_i E_i + B_i B_i + (D_i\phi)(D_i\phi)\right) \ .$$

In the previous equation the integral is over space $\mathbb{R}^3$, $i = 1, 2, 3$ are spatial indices, $E_i = F_{i0}$, $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$, D_i is the gauge covariant derivative, and we do not distinguish between upper and lower spatial indices.

(a) Write the coefficient in front of $(D_i\phi)(D_i\phi)$ as $1 = \sin^2\alpha + \cos^2\alpha$ for a constant angle α and use the Bogomol'nyi trick twice to find a lower bound for the energy.

(b) Use the Bianchi identity $\epsilon_{ijk}D_iF_{jk} = 0$ and the equation of motion $D_iF_{0i} = 0$, which are obeyed by these field configurations, to write the lower bound for the energy as a boundary term.

(c) Maximize the lower bound for the energy.