Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH4061-WE01

Title:
Advanced Quantum Theory IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Consider the free theory of two real scalar fields ϕ_1 and ϕ_2 with Lagrangian density

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi_1\,\partial^\mu\phi_1 - \frac{1}{2}\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2!}m^2\phi_1^2 - \frac{1}{2!}m^2\phi_2^2.$$

(a) Show that the theory has a symmetry under the transformation:

$$\phi_1 \to \phi_1' = (\cos\alpha)\,\phi_1 - (\sin\alpha)\,\phi_2,$$
$$\phi_2 \to \phi_2' = (\sin\alpha)\,\phi_1 + (\cos\alpha)\,\phi_2.$$

(b) Find the conserved current J^μ associated to this symmetry.

Hint: You may use that the current associated to a continuous symmetry is given by

$$J^\mu = -K^\mu + \sum_n \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_n\right)}\Delta\phi_n, \quad \delta\phi_n = \alpha\Delta\phi_n, \quad \delta\mathcal{L} = \alpha\,\partial_\mu K^\mu.$$

(c) Suppose that the interaction term

$$\mathcal{L}_{int} = -\frac{1}{4!}\lambda\phi_1^4 - \frac{1}{4}\kappa\phi_1^2\phi_2^2 - \frac{1}{4!}\lambda\phi_2^4,$$

is added to $\mathcal{L}_0$. For which values of λ and κ is the current J^μ still conserved in the interacting theory?

Q2 Consider the interacting theory of two real scalar fields ϕ_1 and ϕ_2 with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_1\,\partial^\mu\phi_1 - \frac{1}{2}\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2!}m_1^2\phi_1^2 - \frac{1}{2!}m_2^2\phi_2^2 - \frac{1}{4}\kappa\phi_1^2\phi_2^2.$$

(a) Write down the Feynman rules for calculating time-ordered vacuum expectation values in this theory.

(b) Draw each Feynman diagram that contributes to

$$\langle 0|\,T\left\{\hat{\phi}_1(x)\,\hat{\phi}_1(y)\right\}|0\rangle,$$

up to and including quadratic order in the coupling constant, where $\hat{\phi}_1$ is an operator corresponding to the classical field ϕ_1. Determine the symmetry factor of each diagram.

(c) Compute $\langle 0|\,T\left\{\hat{\phi}_1(x)\,\hat{\phi}_2(y)\right\}|0\rangle$ to all orders in the coupling constants, where $\hat{\phi}_1$ and $\hat{\phi}_2$ are operators corresponding to the classical fields ϕ_1 and ϕ_2 respectively.

Q3 Consider a single non-relativistic particle moving on a 1-dimensional line. The motion is described by the generalized coordinate (position) q, and the particle's motion is influenced by the potential

$$V(q) = \frac{1}{2}q^2 + \frac{\lambda}{4!}q^4 .$$

Set the particle's mass to unity, i.e., $m_{\text{particle}} = 1$.

(a) We would like to study the quantum-mechanical motion of this particle using the path-integral formalism.

Write down an expression of the generating functional $Z^{\text{particle}}[J]$ in the presence of the source J that couples to q. You can set $\hbar = 1$ for simplicity.

Define the terms that appear in your expression.

(b) Let $\hat{q}_H(t)$ be the operator corresponding to the particle's position. Express the two- and four-point functions

$$\langle 0| T\{\hat{q}_H(t_1)\hat{q}_H(t_2)\}|0\rangle$$
$$\langle 0| T\{\hat{q}_H(t_1)\hat{q}_H(t_2)\hat{q}_H(t_3)\hat{q}_H(t_4)\}|0\rangle$$

as functional derivatives of $Z^{\text{particle}}[J]$. Do not take the derivatives at this point.

(c) Recall that the action of a single free scalar field with mass m in $d+1$ dimensions is

$$S_0 = -\int d^{d+1}x \left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2\right\}$$

and its generating functional is

$$Z_0[J] = \mathcal{N} \exp\left[\frac{1}{2}\int d^{d+1}x d^{d+1}y J(x) G_F(x,y) J(y)\right] ,$$

where G_F is the Feynman propagator (or Green's function) of ϕ and $\mathcal{N}$ is a normalization factor.

By analogy between the massive scalar field theory and the non-relativistic particle moving in the potential $V(q) = \frac{1}{2}q^2 + \frac{\lambda}{4!}q^4$, write down an expression of generating-functional $Z_0^{\text{particle}}[J]$ of the particle in the absence of the perturbation $\frac{\lambda}{4!}q^4$.

Finally, express the full $Z^{\text{particle}}[J]$ of the particle, in the presence of perturbation, as a formal functional derivative acting on $Z_0^{\text{particle}}[J]$. Do not expand your expression, and do not take the functional derivatives.

Q4 (a) In any given QFT, when do we need to regularize a Feynman diagram?

(b) Consider the following action of $(d+1)$-dimensional scalar field theory, which describes the interaction between two real fields ϕ_1 and ϕ_2

$$S = -\int dx^{d+1} \left\{ \frac{1}{2}\partial_\mu\phi_1\,\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\,\partial^\mu\phi_2 + \frac{\lambda}{4!}(\phi_1 + \phi_2)^4 \right\} .$$

Draw the one-loop Feynman diagram(s) contributing to the two-point function

$$\langle 0|\, T\left\{ \hat{\phi}_1(x_1)\hat{\phi}_1(x_2) \right\} |0\rangle,$$

to first order in λ.

Then, use the momentum-space techniques and introduce a cutoff scale Λ to compute one of these diagrams in any dimension $d+1$.

(c) What is the degree of divergence of this diagram in spacetime dimension $d+1$, assuming $d > 1$?

SECTION B

Q5 Consider the free theory of a complex scalar field ϕ of mass m with Lagrangian density

$$\mathcal{L} = -\left(\partial_\mu \phi^*\right)\left(\partial^\mu \phi\right) - m^2 \phi^* \phi.$$

(a) Explain how to quantise the theory using canonical quantisation.

(b) Upon quantisation, the operator $\hat{\phi}$ corresponding to the classical field ϕ can be expressed in the form

$$\hat{\phi}(t, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left[\hat{a}_{\boldsymbol{k}} \exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega_{\boldsymbol{k}} t\right) + \hat{b}^\dagger_{\boldsymbol{k}} \exp\left(-i\boldsymbol{k}\cdot\boldsymbol{x} + i\omega_{\boldsymbol{k}} t\right)\right],$$

where $\omega_{\boldsymbol{k}}^2 = |\boldsymbol{k}|^2 + m^2$ with $\left[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}\right] = (2\pi)^3\,\delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right)$ and $\left[\hat{b}_{\boldsymbol{k}}, \hat{b}^\dagger_{\boldsymbol{k}'}\right] = (2\pi)^3\,\delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right)$ (and all other commutators of $\hat{a}_{\boldsymbol{k}}$, $\hat{b}_{\boldsymbol{k}}$ and their Hermitian conjugates vanishing).

The conserved charges Q, energy H and spatial momentum P are promoted to the following (normal-ordered) operators:

$$\hat{Q} = \int \frac{d\boldsymbol{k}'}{(2\pi)^3} \left(\hat{a}^\dagger_{\boldsymbol{k}'}\hat{a}_{\boldsymbol{k}'} - \hat{b}^\dagger_{\boldsymbol{k}'}\hat{b}_{\boldsymbol{k}'}\right),$$

$$\hat{H} = \int \frac{d\boldsymbol{k}'}{(2\pi)^3} \omega_{\boldsymbol{k}'} \left(\hat{a}^\dagger_{\boldsymbol{k}'}\hat{a}_{\boldsymbol{k}'} + \hat{b}^\dagger_{\boldsymbol{k}'}\hat{b}_{\boldsymbol{k}'}\right),$$

$$\hat{P} = \int \frac{d\boldsymbol{k}'}{(2\pi)^3} \boldsymbol{k}' \left(\hat{a}^\dagger_{\boldsymbol{k}'}\hat{a}_{\boldsymbol{k}'} + \hat{b}^\dagger_{\boldsymbol{k}'}\hat{b}_{\boldsymbol{k}'}\right).$$

Construct the states that correspond to a single particle of energy ω_k and spatial momentum $\boldsymbol{k}$, showing that they are eigenstates of the appropriate operators. How can these states be distinguished?

(c) Construct an N-particle state with energy $\sum_{i=1}^{N} \omega_{k_i}$ and spatial momentum $\sum_{i=1}^{N} k_i$, showing that it is an eigenstate of the appropriate operators. How many states have this energy and spatial momentum? **Hint:** You can assume that states differing by permutations of the momenta are identical.

Q6 Consider the interacting theory of a real scalar field ϕ with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2!}m^2\phi^2 - \frac{1}{3!}g\phi^3 - \frac{1}{4!}\lambda\phi^4.$$

(a) Write down the Feynman rules for this theory.

(b) Draw all vacuum bubble diagrams up to and including quadratic order in the coupling constants. Give the corresponding expressions in terms of Feynman propagators (you do not need to evaluate any of the integrals).

(c) Draw all Feynman diagrams contributing to the time-ordered vacuum expectation value

$$\langle 0|\, T\left\{\hat{\phi}(x_1)\,\hat{\phi}(x_2)\,\hat{\phi}(x_3)\,\hat{\phi}(x_4)\right\}|0\rangle,$$

up to and including linear order in the coupling constants, where $\hat{\phi}$ is the operator corresponding to the classical field ϕ.

Which of your Feynman diagrams give rise to non-trivial particle scattering? Give the corresponding analytic expression.

Q7 Consider a 4-dimensional complex scalar field ϕ with action

$$S = -\int d^4x \left\{ \partial_\mu \phi^* \partial^\mu \phi + m\phi\phi^* + \frac{\lambda}{3!}(\phi\phi^*)^2 \right\} .$$

When dealing with the complex scalar field theory, one should consider ϕ and ϕ^* as two independent fields.

(a) We can couple the fields ϕ and ϕ^* to the sources J^* and J, respectively, by adding the term $J^*\phi + J\phi^*$ to the above action. Using the change of variables

$$\begin{aligned} \phi &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\,, \quad \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)\,, \\ J &= \frac{1}{\sqrt{2}}(J_1 + iJ_2)\,, \quad J^* = \frac{1}{\sqrt{2}}(J_1 - iJ_2)\,, \end{aligned}$$

where ϕ_1, ϕ_2, J_1, and J_2, are real variables, show that the above action and the source terms $J^*\phi + J\phi^*$ give the expected action and the source terms of two interacting massive real scalar fields with identical masses.

(b) Using the expression of the generating functionals of the free real scalar fields $Z_0[J_1]$ and $Z_0[J_2]$ show that the generating functional of a free complex scalar field is

$$Z_0[J, J^*] = Z_0[J_1]Z_0[J_2] = \mathcal{N} \exp\left[\int d^4x d^4y J^*(x) G_F(x, y) J(y) \right] .$$

(c) Write down an expression of the generating functional of the interacting complex scalar $Z[J, J^*]$ as a functional derivative of the free-field generating functional $Z_0[J, J^*]$ to first order in the coupling constant λ. You should treat J and J^* as two independent sources.

(d) Carry out the functional derivatives of the interaction term using the diagrammatic technique (use a cross for the source J and a square for the source J^*) to find an expression of $Z[J]$ including terms that are first order in λ.

(e) Determine the normalization constant $\mathcal{N}$.

Q8 In this problem, using the light-cone gauge, you will study the motion of a rotating open string in the X^2-X^3 plane. The string motion is defined by $x^- = x^i = 0$, where $I = 2, 3, ..., d$, and the vanishing of all the coefficients α^i_n, with the exception of:

$$\alpha^2_1 = \alpha^{2*}_{-1} = a\,, \quad \alpha^3_1 = \alpha^{3*}_{-1} = ia\,.$$

Above, a is a dimensionless real constant.

(a) What is the mass of this string?

(b) Construct the explicit functions $X^2(\tau, \sigma)$ and $X^3(\tau, \sigma)$.

(c) Calculate the L_n modes for all n.

(d) Find $X^-(\tau, \sigma)$, and show that it can be written solely as a function of p^+, τ and a.

Then, determine the value of p^+ in terms of a and α' using the condition $X^1(\tau, \sigma) = 0$.

Find the relation between X^0 and τ.

The following relations are useful:

$$X^\mu_N = x^\mu + \frac{p^\mu \tau}{\pi T} + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n e^{-in\tau} \cos(n\sigma)\,, \quad 0 \leq \sigma \leq \pi$$

$$\alpha^\mu_0 = \frac{1}{\sqrt{\pi T}} p^\mu$$

$$M^2 = \frac{1}{\alpha'} \sum_{n \geq 1} \alpha^\mu_{-n} \alpha^\nu_n \eta_{\mu\nu}\,, \quad \alpha' = \frac{1}{2\pi T}$$

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n} \alpha^\nu_n \eta_{\mu\nu}$$

$$X^\pm = \frac{X^0 \pm X^1}{\sqrt{2}}$$, **and similar relations hold for** $x^\pm$ *and* $p^\pm$

$$X^+ = 2\alpha' p^+ \tau$$

$$\alpha^-_n = \frac{1}{2\sqrt{2\alpha'} p^+} \sum_{m=-\infty}^{\infty} \alpha^i_{n-m} \alpha^i_m$$