

EXAMINATION PAPER

Examination Session: May/June

2023

Year:

Exam Code:

MATH40820-WE01

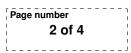
Title:

General Relativity V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.	

Revision:



SECTION A

- **Q1** A two-dimensional spacetime has coordinates (t, x). New local coordinates are defined by $p = e^t$, $q = xe^{-t}$.
 - (a) A vector field V^{μ} has components $V^{\mu} = (1, 1)$ with respect to the original coordinates. Calculate its components \tilde{V}^{μ} with respect to the new coordinates.
 - (b) An antisymmetric (0,2) tensor field $\omega_{\mu\nu}$ (so $\omega_{\mu\nu} = -\omega_{\nu\mu}$) has $\omega_{tx} = 1$ with respect to the original coordinates. Calculate its components $\tilde{\omega}_{\mu\nu}$ with respect to the new coordinates.
- **Q2** In a spacetime with metric $ds^2 = -dt^2 + t^2dx^2 + dy^2 + dz^2$, a curve is defined by $t = \lambda, x = 0, y = \lambda^2, z = \lambda$ for $\lambda \in (1, 2)$.
 - (a) For the vector field $U^{\mu} = (t^2, 2, y 2, 0)$, compute $g_{\mu\nu}U^{\mu}U^{\nu}$. Show that this function is constant along the given curve.
 - (b) Calculate the proper length of this curve.
- Q3 Consider the spacetime with metric

$$ds^{2} = -dt^{2} + \exp(2Ht) \left(dr^{2} + r^{2} \left(d\theta^{2} + \sin^{2}\theta d\phi^{2} \right) \right) ,$$

where H is a constant.

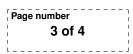
- (a) An astronaut at fixed $r = r_1$, $\theta = \pi/2$, $\phi = 0$, sends a light-ray radially towards r = 0 at coordinate time $t = t_1$. What is the value of the coordinate r of the light-ray at a later time $t > t_1$?
- (b) What frequency would an observer at fixed r = 0, $\theta = \pi/2$, $\phi = 0$, measure for light sent by the astronaut of part a) with frequency ω ?
- **Q4** Consider the following matter action in d spacetime dimensions:

$$S[B_{\mu},g_{\mu\nu}] = \int d^d x \sqrt{-g} \left(\left(\partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} \right) \left(\partial^{\mu} B^{\nu} - \partial^{\nu} B^{\mu} \right) + \alpha B_{\mu} B^{\mu} \right) \,.$$

(a) Evaluate the stress tensor corresponding to this matter action. You can use without proof that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$$

(b) Take $\alpha = 0$. Evaluate T^{μ}_{μ} , does it vanish for any value of d?



SECTION B

Q5 Consider a spacetime with metric

$$ds^{2} = -\mu^{2}r^{2}du^{2} + 2dudv + dr^{2} + r^{2}d\phi^{2},$$

where μ is a constant.

- (a) Calculate the Christoffel symbols for this metric.
- (b) This metric has three Killing vectors. Write down the corresponding conserved quantities for geodesics in this metric.
- (c) Show that geodesics with ϕ constant and u not constant have $r = \alpha \sin \beta \lambda$, where λ is an affine parameter along the geodesic, and α and β are constants, which you should relate to the conserved quantities identified in the previous part.
- (d) Give expressions for $u(\lambda)$ and $v(\lambda)$ for these geodesics. [You may find the integral $\int \sin^2 x dx = \frac{1}{2}(x \sin x \cos x)$ useful.]
- Q6 (a) Consider a spacetime with metric $g_{\mu\nu} = \exp(2f(x^{\mu}))\eta_{\mu\nu}$, where $f(x^{\mu})$ is a scalar function and $\eta_{\mu\nu}$ is the Minkowski metric. Calculate the Christoffel symbols for this metric, and write the geodesic equation for null geodesics. Show that there is a non-affine choice of parameter σ such that the null geodesic equation can be rewritten as $d^2x^{\mu}/d\sigma^2 = 0$.
 - (b) Suppose that a scalar field ϕ satisfies $\nabla_{\lambda} \nabla^{\lambda} \phi = 0$. Show that the tensor

$$U_{\mu\nu} = \nabla_{\mu}\nabla_{\nu}\phi - \phi R_{\mu\nu}$$

satisfies $\nabla_{\nu} U^{\mu\nu} = A \phi \nabla^{\mu} R$, where A is a constant you should compute.

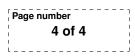
Q7 Consider the metric

$$ds^{2} = -(1+r^{2})dt^{2} + \frac{dr^{2}}{(1+r^{2})} + r^{2}\left(d\theta^{2} + \sin^{2}\theta d\phi^{2}\right) \,.$$

- (a) A photon is emitted at r = t = 0 and travels radially outwards to r = L. At which coordinate time does it reach r = L? Is this time finite as $L \to \infty$?
- (b) How much time would an observer at r = 3, $\theta = \pi/2$, $\phi = 0$ measure for the time it takes the photon of part a) to reach r = L?
- (c) Write down the geodesic equation for a massive particle in this spacetime as an equation for the derivative of r with respect to the proper time in terms of constants of motion for the path and r. You may take the motion to be in the $\theta = \pi/2$ plane. Is the observer of part b) following a geodesic?

You may need the following integral

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) \,.$$





Q8 Consider the spacetime

$$ds^{2} = -dt^{2} + dr^{2} + r^{2}(1-\alpha)^{2}d\phi^{2},$$

where $0 \leq \phi < 2\pi$ and r are polar coordinates and $\alpha < 1$ is a positive constant.

- (a) Are there any singularities (coordinate or curvature)? Justify your answer. You do not need to classify the type of singularity.
- (b) Can a spaceship take the path r(t) = 3t, $\phi(t) = 0$? Justify your answer.
- (c) Find a change of variables, to variables \bar{r} and $\bar{\phi}$, where the metric looks like

$$ds^2 = -dt^2 + d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2$$

Is this flat space? Describe what this spacetime looks like.

(d) Consider the spatial part of the metric. Is this space homogenous for $\alpha \neq 0$? Is it isotropic about r = 0?