EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH40820-WE01

Title:
General Relativity V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 A two-dimensional spacetime has coordinates (t, x). New local coordinates are defined by $p = e^t$, $q = xe^{-t}$.

(a) A vector field V^μ has components $V^\mu = (1, 1)$ with respect to the original coordinates. Calculate its components $\tilde{V}^\mu$ with respect to the new coordinates.

(b) An antisymmetric $(0, 2)$ tensor field $\omega_{\mu\nu}$ (so $\omega_{\mu\nu} = -\omega_{\nu\mu}$) has $\omega_{tx} = 1$ with respect to the original coordinates. Calculate its components $\tilde{\omega}_{\mu\nu}$ with respect to the new coordinates.

Q2 In a spacetime with metric $ds^2 = -dt^2 + t^2dx^2 + dy^2 + dz^2$, a curve is defined by $t = \lambda, x = 0, y = \lambda^2, z = \lambda$ for $\lambda \in (1, 2)$.

(a) For the vector field $U^\mu = (t^2, 2, y - 2, 0)$, compute $g_{\mu\nu}U^\mu U^\nu$. Show that this function is constant along the given curve.

(b) Calculate the proper length of this curve.

Q3 Consider the spacetime with metric

$$ds^2 = -dt^2 + \exp(2Ht)\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) ,$$

where H is a constant.

(a) An astronaut at fixed $r = r_1$, $\theta = \pi/2$, $\phi = 0$, sends a light-ray radially towards $r = 0$ at coordinate time $t = t_1$. What is the value of the coordinate r of the light-ray at a later time $t > t_1$?

(b) What frequency would an observer at fixed $r = 0$, $\theta = \pi/2$, $\phi = 0$, measure for light sent by the astronaut of part a) with frequency ω?

Q4 Consider the following matter action in d spacetime dimensions:

$$S[B_\mu, g_{\mu\nu}] = \int d^dx\sqrt{-g}\left(\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)\left(\partial^\mu B^\nu - \partial^\nu B^\mu\right) + \alpha B_\mu B^\mu\right) .$$

(a) Evaluate the stress tensor corresponding to this matter action. You can use without proof that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$$

(b) Take $\alpha = 0$. Evaluate T^μ_μ, does it vanish for any value of d?

SECTION B

Q5 Consider a spacetime with metric

$$ds^2 = -\mu^2 r^2 du^2 + 2dudv + dr^2 + r^2 d\phi^2,$$

where μ is a constant.

(a) Calculate the Christoffel symbols for this metric.

(b) This metric has three Killing vectors. Write down the corresponding conserved quantities for geodesics in this metric.

(c) Show that geodesics with ϕ constant and u not constant have $r = \alpha \sin \beta\lambda$, where λ is an affine parameter along the geodesic, and α and β are constants, which you should relate to the conserved quantities identified in the previous part.

(d) Give expressions for $u(\lambda)$ and $v(\lambda)$ for these geodesics. [You may find the integral $\int \sin^2 x dx = \frac{1}{2}(x - \sin x \cos x)$ useful.]

Q6 (a) Consider a spacetime with metric $g_{\mu\nu} = \exp(2f(x^\mu))\eta_{\mu\nu}$, where $f(x^\mu)$ is a scalar function and $\eta_{\mu\nu}$ is the Minkowski metric. Calculate the Christoffel symbols for this metric, and write the geodesic equation for null geodesics. Show that there is a non-affine choice of parameter σ such that the null geodesic equation can be rewritten as $d^2x^\mu/d\sigma^2 = 0$.

(b) Suppose that a scalar field ϕ satisfies $\nabla_\lambda \nabla^\lambda \phi = 0$. Show that the tensor

$$U_{\mu\nu} = \nabla_\mu \nabla_\nu \phi - \phi R_{\mu\nu}$$

satisfies $\nabla_\nu U^{\mu\nu} = A\phi \nabla^\mu R$, where A is a constant you should compute.

Q7 Consider the metric

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{(1+r^2)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) .$$

(a) A photon is emitted at $r = t = 0$ and travels radially outwards to $r = L$. At which coordinate time does it reach $r = L$? Is this time finite as $L \to \infty$?

(b) How much time would an observer at $r = 3$, $\theta = \pi/2$, $\phi = 0$ measure for the time it takes the photon of part a) to reach $r = L$?

(c) Write down the geodesic equation for a massive particle in this spacetime as an equation for the derivative of r with respect to the proper time in terms of constants of motion for the path and r. You may take the motion to be in the $\theta = \pi/2$ plane. Is the observer of part b) following a geodesic?

You may need the following integral

$$\int \frac{dx}{1+x^2} = tan^{-1}(x)\,.$$

Q8 Consider the spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(1-\alpha)^2 d\phi^2\,,$$

where $0 \leqslant \phi < 2\pi$ and r are polar coordinates and $\alpha < 1$ is a positive constant.

(a) Are there any singularities (coordinate or curvature)? Justify your answer. You do not need to classify the type of singularity.

(b) Can a spaceship take the path $r(t) = 3t$, $\phi(t) = 0$? Justify your answer.

(c) Find a change of variables, to variables $\bar{r}$ and $\bar{\phi}$, where the metric looks like

$$ds^2 = -dt^2 + d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2\,.$$

Is this flat space? Describe what this spacetime looks like.

(d) Consider the spatial part of the metric. Is this space homogenous for $\alpha \neq 0$? Is it isotropic about $r = 0$?