

EXAMINATION PAPER

Examination Session: May/June

2023

Year:

Exam Code:

MATH40920-WE01

Title:

Mathematical Finance V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:



SECTION A

- **Q1** Consider the market consisting of one risk-free asset with price dynamics $B_t = (6/5)^t$, t = 0, 1, 2, and one risky asset whose price S_t evolves with $S_0 = 15$, u = 1.4 and d = 0.6.
 - (a) Does this market contain an opportunity for arbitrage?
 - (b) A broker offers a European put option on the risky asset with expiry date T = 2 and strike price K = 15. Construct a hedging portfolio for the option.
 - (c) The option can be bought for £1. Is this a fair price? Either justify that it is, or describe how to construct a portfolio involving the option which returns a guaranteed profit.

- **Q2** (a) State the definition of *Brownian motion*;
 - (b) Let $(W_t)_{t\geq 0}$ be a Brownian motion and a > 0. Prove that

$$\left(\frac{1}{\sqrt{a}}W_{at}\right)_{t\geq 0}$$
 is a Brownian motion;

- (c) State the definition for a stochastic process $(X_t)_{t\geq 0}$ to be a martingale with respect to a given filtration $(\mathcal{F}_t)_{t\geq 0}$;
- (d) Let $(W_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by $(W_t)_{t\geq 0}$. Prove that

 $(W_t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Q3 Let $(W_t)_{t\geq 0}$ be a Brownian motion.

- (a) State the rules of Itô calculus (box calculus) among dt and d W_t , and state also the Itô formula for $dF(t, S_t)$ with a smooth function F(t, x) and Itô process $(S_t)_{t \ge 0}$;
- (b) Apply the Itô formula to $d(W_t^2)$ and express $d(W_t^2)$ in terms of dt, dW_t and W_t ;
- (c) Prove

$$\int_0^t W_s dW_s = \frac{1}{2} (W_t^2 - t) \; .$$



SECTION B

- Q4 (a) Give the definitions of American call and put options.
 - (b) Given a (general) European claim, and the corresponding American claim, is one worth more than the other? Why?
 - (c) Consider the binomial market model with 3 periods, in which $S_0 = 100$, and the share price evolves with u = 1.2 and d = 0.8. Suppose that the risk-free interest rate is r = 0.1. What is the fair price at time 0 for an American put option with strike price K = 100? Under which circumstances is it wise to exercise the option early?
- **Q5** (a) What does it mean for a random variable $X : \Omega \to \mathbb{R}$ to be *measurable* with respect to a σ -algebra \mathcal{F} ?
 - (b) A random variable τ is a *stopping time* with respect to a discrete-time filtration \mathcal{F}_n if, for every n, the indicator random variable $\mathbb{1}\{\tau \leq n\}$ is measurable with respect to \mathcal{F}_n . In the context of the multi-period binomial model, give an example of:
 - (i) a random variable that *is* a stopping time
 - (ii) a random variable that *is not* a stopping time.
 - (c) Show that the fair price for an American put option, P^A , with expiry date T, strike price K and interest rate r, satisfies

$$K(1+r)^{-T} - S_0 \le P^A \le K(1+r)^{-T},$$

whatever happens to the share prices.





Q6 Let $B_t = e^{rt}$ be the bond price with interest rate r > 0 and $(S_t)_{t \ge 0}$ be the stock price following the Black-Scholes-Merton SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \mu \in \mathbb{R}, \quad \sigma > 0, \quad S_0 = x_0 > 0, \quad t \in [0, T] .$$

Let $(\mathcal{F}_t)_{t \in [0,T]}$ be the filtration generated by $(W_t)_{t \in [0,T]}$.

- (a) Prove $dS_t dS_t = \sigma^2 S_t^2 dt$;
- (b) Let (a_t, b_t) be a portfolio and $V_t = a_t B_t + b_t S_t$ be the value process. State the definition of the self-financing condition;
- (c) Let $\Phi = \Phi(S_T)$ be a contingent claim with expiry time T. State the definition of the portfolio (a_t, b_t) replicating Φ ;
- (d) Let $\Pi_t(\Phi)$ be the fair price (no-arbitrage price) of the contingent claim Φ at t, and let $F(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a smooth function so that $F(t, S_t) = \Pi_t(\Phi)$. Prove that F(t, x) is given by the following partial differential equation:

$$\begin{cases} \partial_t F + \frac{1}{2}\sigma^2 x^2 \partial_{xx}F + rx \partial_x F - rF = 0\\ F(T, x) = \Phi(x) \ . \end{cases}$$

Q7 Let $B_t = e^{rt}$ be the bond price with interest rate r > 0 and $(S_t)_{t \ge 0}$ be the stock price following the Black-Scholes-Merton SDE:

 $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad \mu \in \mathbb{R}, \quad \sigma > 0, \quad S_0 = x_0 > 0, \quad t \in [0, T] \; .$

Let $(\mathcal{F}_t)_{t \in [0,T]}$ be the filtration generated by $(W_t)_{t \in [0,T]}$. Let $(\tilde{W}_t)_{t \ge 0}$ be the Brownian motion under the risk-neutral measure (martingale measure) \mathbb{Q} .

- (a) Write down the formula expressing $(W_t)_{t\geq 0}$ in terms of $(W_t)_{t\geq 0}$, μ , σ , r, t;
- (b) Prove that the process $\tilde{S}_t := e^{-rt} S_t$ is martingale with respect to \mathbb{Q} ;
- (c) State Martingale Representation Theorem;
- (d) Let Φ be a contingent claim with expiry time T. Let $\Pi_t(\Phi)$ be the fair price (no-arbitrage price) of the contingent claim Φ at t. Prove

$$\Pi_t(\Phi) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[\Phi|\mathcal{F}_t].$$





SECTION C

- **Q8** (a) Suppose that $(S_t, t \in [0, T])$ is the price of a risky asset that evolves randomly according to a geometric Brownian motion with drift μ and volatility σ . Suppose that interest is compounded continuously at rate r. Carefully describe a Monte Carlo algorithm to produce an approximate 95% confidence interval for the no-arbitrage price at time $t \in [0, t]$ of a contingent claim with payoff $F(S_T)$ from a sample of M standard Normal random variables. You should explain any notation you use in your answer.
 - (b) Describe two ways in which the size of the confidence interval produced by your Monte Carlo approximation may be reduced.