Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH41120-WE01

Title:
Algebraic Topology V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

The following conventions hold in this paper:
The n-dimensional sphere with the Euclidean topology is denoted by S^n.
The closed n-dimensional unit disc with the Euclidean topology is denoted by D^n.

SECTION A

Q1 Let X be a non-empty topological space.

(a) State the definition of the cone of X, denoted CX.

(b) Show that CX is contractible.

(c) State the definition of the suspension of X, denoted SX.

(d) Show that if X is contractible, then SX is contractible.

Q2 Let $H_k(X)$ denote the k-th singular homology group of a topological space X with coefficients in $\mathbb{Z}$.

(a) State the definition of the degree of a map $f\colon S^n \to S^n$ with $n \geq 1$.

(b) Construct a CW-complex X with

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/3 & k = 2, \\ \mathbb{Z}/9\mathbb{Z} & k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Justify your construction. You can assume the existence of maps of specific degree.

Q3 In this exercise we denote by $\mathbb{RP}^3$ the 3-dimensional real projective space.

(a) Describe $\mathbb{RP}^3$ as a CW-complex, deduce the complex underlying its CW-homology, and compute its homology.

(b) Provide a compact, 4-dimensional manifold with non-empty boundary W which is homotopy-equivalent to $Y = \mathbb{RP}^3$. Describe the homotopy-equivalence explicitly.

(c) Is there a closed, connected 4-dimensional oriented manifold X which is homotopy-equivalent to $\mathbb{RP}^3$?

Q4 Let M be an n-dimensional connected, oriented manifold with non-empty connected boundary ∂M, and let $n \geq 2$. The boundary ∂M is itself an orientable manifold (you do not need to prove this).

(a) State the Poincaré-Lefschetz duality theorem which holds in this situation.

(b) Compute $H_n(M)$.

(c) In this exercise you can use that the relative cohomology group $H^1(M, \partial M; \mathbb{Z})$ has no torsion, without proving it. Conclude that in the long exact homology sequence of the pair $(M, \partial M)$ the connecting homomorphism $H_n(M, \partial M) \to H_{n-1}(\partial M)$ is an isomorphism.

SECTION B

Q5 (a) Let (C_*, ∂_*) be a chain complex over $\mathbb{Z}$. Assume that there exist homomorphisms $H_n\colon C_n \to C_{n+1}$ with $H_{n-1} \circ \partial_n + \partial_{n+1} \circ H_n = \mathrm{id}_{C_n}$ for all $n \in \mathbb{Z}$. Show that $H_n(C_*) = 0$ for all $n \in \mathbb{Z}$.

(b) Consider the chain complex (C_*, ∂_*) where $C_n = 0$ for $n < 0$ or $n > 4$, and the remaining chain groups fit in a sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \mathbb{Z} \longrightarrow 0$$

Show that $H_0\colon C_0 \to C_1$ given by

$$\mathbb{Z} \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}$$

can be extended to homomorphisms $H_n\colon C_n \to C_{n+1}$ with

$$H_{n-1} \circ \partial_n + \partial_{n+1} \circ H_n = \mathrm{id}_{C_n}$$

for all $n \in \mathbb{Z}$.

(c) Consider the chain complex (D_*, ∂_*) given by $D_n = \mathbb{Z}/4\mathbb{Z}$ and $\partial_n\colon D_n \to D_{n-1}$ given by multiplication with $2 \in \mathbb{Z}/4\mathbb{Z}$ for every $n \in \mathbb{Z}$. Calculate $H_n(D_*)$ for every $n \in \mathbb{Z}$.

(d) Decide whether there exists $H_n\colon D_n \to D_{n+1}$ with $H_{n-1} \circ \partial_n + \partial_{n+1} \circ H_n = \mathrm{id}_{D_n}$ for all $n \in \mathbb{Z}$. Justify your statement.

Q6 In this question $H_k(X)$ is the k-th singular homology group of a topological space X with coefficients in $\mathbb{Z}$. Let $f\colon S^1 \to S^1$ be a map of degree 2. For n a positive integer let

$$T_n = \left(\coprod_{i=0}^{n} S^1 \times [i, i+1] \right) \Big/ \sim,$$

where $(z, i+1) \in S^1 \times [i, i+1]$ is identified by $\sim$ with $(f(z), i+1) \in S^1 \times [i+1, i+2]$ for all $i = 0, \ldots, n-1$.

(a) Let $i_n\colon T_n \to T_{n+1}$ be the function given by $i_n([z,t]) = [z,t]$ for $[z,t] \in T_n$. Show that i_n is an embedding (that is, an injective map which is a homeomorphism onto its image) of T_n into T_{n+1}.

(b) Calculate $H_k(T_n)$ for all $k \geq 0$ and $n \geq 1$, and determine the homomorphism $i_{n*}\colon H_k(T_n) \to H_k(T_{n+1})$ for all $n, k \geq 1$.

(c) We now treat $T_n \subset T_{n+1}$ using i_n from the previous question. Let

$$T = \bigcup_{n=1}^{\infty} T_n$$

and topologise T as follows: A subset $A \subset T$ is closed if and only if $A \cap T_n$ is closed in T_n for all $n \geq 1$. Show that the set-theoretic inclusion $j_n\colon T_n \to T$ is an embedding for all $n \geq 1$.

(d) Show that $j_{n*}\colon H_1(T_n) \to H_1(T)$ is injective, and $H_1(T)$ does not contain torsion elements. Furthermore, show that for each $x \in H_1(T)$ there exists $y \in H_1(T)$ with $2y = x$.

Hint: You may use the following result from the lectures without proof: Let X be a topological space and $(U_n)_{n\in\mathbb{N}}$ open sets with $U_n \subset U_{n+1}$ and $X = \bigcup_{n=1}^{\infty} U_n$. Then

- For each $\alpha \in H_k(X)$ there is $n \in \mathbb{N}$ with $\alpha \in \mathrm{im}(j_{n*}\colon H_k(U_n) \to H_k(X))$.
- If $\alpha_n \in H_k(U_n)$ satisfies $j_{n*}\alpha_n = 0 \in H_k(X)$, then there is $m > n$ with $j_{n,m*}\alpha_n = 0 \in H_k(U_m)$, where $j_{n,m}\colon U_n \to U_m$ is inclusion.

Q7 Consider the sequence of abelian groups

$$0 \to C_2 := \mathbb{Z}/6 \xrightarrow{\partial_2 := \times 2} C_1 := \mathbb{Z}/6 \xrightarrow{\partial_1 := \times 3} C_0 := \mathbb{Z}/6 \to 0,$$

where $\times 2\colon \mathbb{Z}/6 \to \mathbb{Z}/6$ denotes the map induced from multiplication with 2 on $\mathbb{Z}$, and likewise $\times 3\colon \mathbb{Z}/6 \to Z/6$ denotes the map induced from multiplication with 3 on $\mathbb{Z}$.

(a) Show that this sequence is a complex of abelian groups (C_*, ∂_*).

(b) Determine its homology groups $H_*(C_*)$.

(c) Determine the cohomology groups $H^*(C^*; \mathbb{Z}/6)$ with coefficients in $\mathbb{Z}/6$.

(d) State the groups $\mathrm{Ext}^1(\mathbb{Z}/2, \mathbb{Z}/6)$ and $\mathrm{Ext}^1(\mathbb{Z}/3, \mathbb{Z}/6)$. (It is not required to derive these using a resolution.)

(e) Explain why the homology groups $H_*(C_*)$ and $H^*(C^*; \mathbb{Z}/6)$ cannot arise as singular homology $H_*(X)$ and singular cohomology $H^*(X; \mathbb{Z}/6)$ for a finite CW-complex X.

Q8 The aim of this problem is to conclude that there is no continuous map

$$f\colon S^2 \times S^2 \to \mathbb{CP}^2$$

of degree ± 1. In this exercise M will always denote a n-dimensional closed connected oriented manifold for some integer $n \geq 2$, and we denote by $[M] \in H_n(M; \mathbb{Z})$ its fundamental class.

(a) Show that $H^n(M; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

(b) Show that the evaluation map $\mathrm{ev}\colon H^n(M; \mathbb{Z}) \to \mathrm{Hom}(H_n(M), \mathbb{Z})$ is an isomorphism.

(c) Conclude that there is a unique element $[M]^* \in H^n(M; \mathbb{Z})$ such that one has $\langle [M]^*, [M] \rangle = 1$. This is called the dual fundamental class.

(d) Suppose $f\colon M \to N$ is a map of degree $\deg(f) = d \in \mathbb{Z}$, where N is a n-dimensional closed connected oriented manifold. Show that one has

$$f^*([N]^*) = d \cdot [M]^*.$$

(e) Show that there is no map $g\colon S^4 \to \mathbb{CP}^2$ of non-zero degree d. (Hint: Use the fact that g^* is a ring homomorphism.)

(f) Show that for any element $a \in H^2(S^2 \times S^2; \mathbb{Z})$ the expression

$$\langle a \smile a, [S^2 \times S^2] \rangle$$

is an even integer.

(g) Conclude that there is no map $f\colon S^2 \times S^2 \to \mathbb{CP}^2$ of odd degree (and in particular not of degree ± 1).