

EXAMINATION PAPER

Examination Session: May/June

2023

Year:

Exam Code:

MATH41220-WE01

Title:

Analysis V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:



SECTION A

- **Q1** (a) (i) Let ϕ be a simple measurable function taking only nonnegative values. State the definition of the integral $\int \phi$ of ϕ .
 - (ii) Let f be a measurable function taking values in the nonnegative extended reals. State the definition of the integral $\int f$ of f.
 - (b) State the Monotone Convergence Theorem.
 - (c) Let f, g be measurable functions taking values in the nonnegative extended reals. Show that if $f \leq g$ then $\int f \leq \int g$. (Hint: you may use that for nonnegative simple measurable functions $\phi \leq \psi$ implies $\int \phi \leq \int \psi$.)
- **Q2** 2.1 Let $E \subset \mathbb{R}$ be measurable and let $1 \leq p < \infty$.
 - (a) State the definition of $L^p(E)$.
 - (b) Prove that if $f \in L^p(E)$ and $E_{\alpha} = \{x \in E : |f(x)| \ge \alpha\}, \alpha > 0$, then $\mu(E_{\alpha}) \to 0$ as $\alpha \to \infty$ (where μ denotes the Lebesgue measure).
 - **2.2** Let q > 1. If $f \in L^q[0, 1]$, then is it true that $f \in L^r[0, 1]$ for $1 \le r < q$? Give a full justification of your response.
- **Q3** Let \mathcal{H} be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let the norm derived from $\langle \cdot, \cdot \rangle$ be denoted by $\|\cdot\|$.
 - (a) State Bessel's Inequality for an orthonormal set $U = \{u_{\alpha} : \alpha \in I\} \subset \mathcal{H}$.
 - (b) Suppose that for any $x \in \mathcal{H}$, $x = \sum_{\alpha \in I} \langle x, u_{\alpha} \rangle u_{\alpha}$ where the sum has at most countably many non-zero terms and converges unconditionally in \mathcal{H} . Prove Parseval's Identity.



SECTION B

Q4 Let $C \subset [0, 1]$ be the middle-third Cantor set.

- (a) Explain why C is a Borel set.
- (b) Show that C has Lebesgue measure 0.
- (c) State what it means for a point x to be an accumulation point of a set $E \subset \mathbb{R}$.
- (d) Define the function $g: [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} x & \text{if } x \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Is g Riemann integrable? Justify your answer.

Q5 In the following $A, B, B_n, E, E_n \subset \mathbb{R}$. Recall that the symmetric difference of two sets A and B is

$$A \Delta B = (A - B) \cup (B - A).$$

- (a) Show that outer measure is monotonic: namely if $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$.
- (b) Show that if $\mu^*(E \Delta B_n) \to 0$ as $n \to \infty$ then $\mu^*(B_n) \to \mu^*(E)$ as $n \to \infty$.
- (c) Suppose that E_n is a finite union of intervals and suppose that $\mu^*(E \Delta E_n) \to 0$ as $n \to \infty$. Show that we have

$$\mu^*(A) = \mu^*(E \cap A) + \mu^*(E^c \cap A).$$

Q6 Let $E \subseteq \mathbb{R}$ be measurable. Let $1 \leq p, q < \infty$. Consider the linear space $L^p(E) \times L^q(E)$ where addition and scalar multiplication are defined pointwise as $(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2)$ and $\lambda(f, g) = (\lambda f, \lambda g)$ for $\lambda \in \mathbb{C}$. Define $\|(\cdot, \cdot)\| : L^p(E) \times L^q(E) \to \mathbb{R}_{\geq 0}$ as

$$||(f,g)|| = (||f||_{L^p}^2 + ||g||_{L^q}^2)^{1/2}, \quad (f,g) \in L^p(E) \times L^q(E).$$

- **6.1** Prove that $\|(\cdot, \cdot)\|$ is a norm on $L^p(E) \times L^q(E)$ for $1 \le p, q < \infty$.
- **6.2** Prove that $(L^p(E) \times L^q(E), ||(\cdot, \cdot)||)$ is a Banach space for $1 \le p, q < \infty$.
- **6.3** Using your response to **6.2** or otherwise, state conditions that normed linear spaces X, Y, with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ respectively, must satisfy to ensure that $X \times Y$ with norm $\|(x, y)\| = (\|x\|_X^2 + \|y\|_Y^2)^{1/2}$ is a Banach space.
- **6.4** Give an example of normed linear spaces $(X, \|\cdot\|_1), (Y, \|\cdot\|_2)$, such that $X \times Y$ with norm $\|(x, y)\| = (\|x\|_1^2 + \|y\|_2^2)^{1/2}$ is not a Banach space. Briefly justify your response.
- **Q7** 7.1 Consider $(L^3[0,1], \|\cdot\|_{L^3})$. Is the norm $\|\cdot\|_{L^3}$ derived from an inner product? Justify your response.
 - **7.2** Let $E \subset \mathbb{R}$ be measurable. Let $f \in L^1(E)$. Prove that

$$\lim_{n \to \infty} \int_E f(x)(\cos(nx))^2 = \frac{1}{2} \int_E f(x).$$

You may use without proof that since $f \in L^1(E)$, for any $\epsilon > 0$, there exists a step function ψ such that $||f - \psi||_{L^1} < \epsilon$. You should otherwise give a full justification.



SECTION C

Q8 In this question we consider the two measure spaces

$$(X, \mathcal{X}, \nu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu), (Y, \mathcal{Y}, m) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu);$$

namely two copies of the real numbers equipped with Lebesgue measure and the Borel σ -algebra. We denote π the product measure associated to these two spaces. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function

$$f(x,y) = \begin{cases} 1 & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

- 8.1 (a) State the definition of the product σ -algebra.
 - (b) Explain why f is measurable.
- **8.2** Denote $f_x(y) = f(x, y)$.
 - (a) Show that

$$\int f_x \, d\mu = \begin{cases} 0 & \text{if } x \le 0, \\ 1 - x & \text{if } 0 < x < 1, \\ 0 & \text{if } x \ge 1. \end{cases}$$

- (b) State Tonelli's Theorem.
- (c) Evaluate, with justification, the integral $\int f d\pi$. (You may use without proof that $\int_{[0,1]} h d\mu = 1/2$ for h(x) = x.)