

EXAMINATION PAPER

Examination Session: May/June

2023

Year:

Exam Code:

MATH41420-WE01

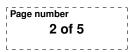
Title:

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Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:



SECTION A

- Q1 Show that for each of the following two equations and boundary conditions, the quantity $\int_{-\infty}^{+\infty} dx \ \rho(u, u_x, u_t)$ is constant:
 - (a) $u_t + u^3 u_x + u_{xxx} = 0$, with $u, u_x, u_{xx} \to 0$ as $x \to \pm \infty$, and $\rho = u^2$;
 - (b) $u_{tt} u_{xx} + \sin u = 0$, with $u, u_x, u_t \to 0$ as $x \to -\infty, u \to -4\pi, u_x, u_t \to 0$ as $x \to +\infty$, and $\rho = u_t u_x$.
- **Q2** (a) If F[u] is the functional $F[u] = \int_{-\infty}^{\infty} dx f(u, u_x, u_{xx}, \ldots)$, define the functional derivative $\delta F[u] / \delta u$ and derive an expression for this derivative in terms of $\partial f / \partial u, \partial f / \partial u_x, \partial f / \partial u_{xx}$ etc.

You may assume that u satisfies the boundary conditions $u \to 0$, $u_x \to 0$, $u_{xx} \to 0$ etc as $x \to \pm \infty$.

(b) Consider $f(u, u_x) = a_1 u_x^2 + a_2 u^3$ and write out the differential equation

$$u_t = \frac{\partial}{\partial x} \left(\frac{\delta F\left[u \right]}{\delta u} \right).$$

Find the values of the constants a_1 and a_2 for which this becomes the KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

Q3 The Marchenko equation is

$$K(x,z) + F(x+z) + \int_{-\infty}^{x} dy K(x,y) F(y+z) = 0.$$

Given $F(x) = -e^{-x}\theta(x)$, show that $K(x, z) = \alpha\theta(x + z)$ solves the Marchenko equation for $z \leq x$ for some constant α that you should determine. Hence find the potential V(x) = 2dK(x, x)/dx.

[The function $\theta(x)$ is the Heaviside step function defined as $\theta(x) = 1$ for x > 0 and $\theta(x) = 0$ for $x \le 0$. Its derivative is the Dirac delta function: $d\theta(x)/dx = \delta(x)$.]

SECTION B

Q4 Consider the pair of equations

$$u_x = -uv_x - e^v$$
$$u_t = uv_t - e^{-v}.$$

Show that these relations give a Bäcklund transform between an equation for u and an equation for v, that you should find.

Q5 Hirota's bilinear differential operators $D_x^m D_t^n$ are defined by

$$(D_x^m D_t^n(f,g))(x,t) := (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x',t') \big|_{x'=x, t'=t}.$$

By setting $u = \log f$ in the differential equation

$$u_{xt} = e^{-u} - e^{-2u} ,$$

show that this equation can be cast into Hirota's bilinear form as

$$D_x D_t(f, f) = H(f) ,$$

where H is a function that you should find. Verify that

$$f = 1 + \exp(ax + bt + c)$$

is a solution, provided that the constants a, b, c satisfy appropriate relations that you should find. What is the velocity of this solution?

Q6 Consider u(x,t) that satisfies the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

(a) If D = d/dx and f(x) is a general function of x, show that

$$[D, f] = f_x,$$

$$[D^2, f] = f_{xx} + 2f_x D.$$

(b) Consider operators

$$L = D^{2} + u(x, t), B = -4D^{3} + \alpha_{1}(x) D + \alpha_{0}(x),$$

Find the functions $\alpha_1(x)$ and $\alpha_0(x)$ of u and u_x such that the KdV equation can be written in the form

$$L_t + [L, B] = 0.$$

You may use that: $[D^3, f] = f_{xxx} + 3f_{xx}D + 3f_xD^2$ for any function f = f(x), together with the formulas in part 6(a).



Q7 Consider the Schrödinger equation

$$\psi'' + \left(k^2 - V\left(x\right)\right)\psi = 0,$$

with potential

$$V(x) = \lambda_1 \delta(x+a) - \lambda_2 \delta(x-a),$$

where $\lambda_1, \lambda_2 > 0$.

(a) By imposing the appropriate matching conditions, derive the equation for bound state solutions $k = i\mu$ with $\mu > 0$ in the form

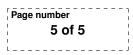
$$(b+2\mu)(c-2\mu) - \lambda_1 \lambda_2 e^{\mu d} = 0,$$

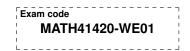
giving the constants b, c and d.

- (b) Find the bound state solutions (if any) as:
 - (i) $a \rightarrow 0$
 - (ii) $a \to \infty$

In each case give the corresponding constraints (if any) on λ_1 and λ_2 .

(c) Taking for simplicity $\lambda_1 = \lambda_2$ and a > 0, is it possible to conclude that there always exists a bound state?





SECTION C

Q8 A field u(x,t) with $x, t \in \mathbb{R}$ is described by the Lagrangian density $\mathcal{L}_L(u, u_x, u_t)$ if x < 0, the Lagrangian density $\mathcal{L}_R(u, u_x, u_t)$ if x > 0, and the Lagrangian $I(u, u_t)|_{x=0}$ at the interface x = 0 between the two regions. The field u(x,t) is continuous at x = 0. Use the variational principle (or principle of stationary action) to find the equations obeyed by the field if x < 0, x > 0 and x = 0 respectively. Under which conditions is the equation of motion at the interface x = 0 independent of the bulk Lagrangian densities at $x \neq 0$?