Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH41420-WE01

Title:
Solitons V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Show that for each of the following two equations and boundary conditions, the quantity $\int_{-\infty}^{+\infty} dx\ \rho(u, u_x, u_t)$ is constant:

(a) $u_t + u^3 u_x + u_{xxx} = 0$, with $u, u_x, u_{xx} \to 0$ as $x \to \pm\infty$, and $\rho = u^2$;

(b) $u_{tt} - u_{xx} + \sin u = 0$, with $u, u_x, u_t \to 0$ as $x \to -\infty$, $u \to -4\pi, u_x, u_t \to 0$ as $x \to +\infty$, and $\rho = u_t u_x$.

Q2 (a) If $F[u]$ is the functional $F[u] = \int_{-\infty}^{\infty} dx\, f(u, u_x, u_{xx}, \ldots)$, define the functional derivative $\delta F[u]/\delta u$ and derive an expression for this derivative in terms of $\partial f/\partial u$, $\partial f/\partial u_x$, $\partial f/\partial u_{xx}$ etc.

You may assume that u satisfies the boundary conditions $u \to 0$, $u_x \to 0$, $u_{xx} \to 0$ etc as $x \to \pm\infty$.

(b) Consider $f(u, u_x) = a_1 u_x^2 + a_2 u^3$ and write out the differential equation

$$u_t = \frac{\partial}{\partial x}\left(\frac{\delta F[u]}{\delta u}\right).$$

Find the values of the constants a_1 and a_2 for which this becomes the KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

Q3 The Marchenko equation is

$$K(x, z) + F(x + z) + \int_{-\infty}^{x} dy\, K(x, y)\, F(y + z) = 0.$$

Given $F(x) = -e^{-x}\theta(x)$, show that $K(x, z) = \alpha\theta(x + z)$ solves the Marchenko equation for $z \leq x$ for some constant α that you should determine. Hence find the potential $V(x) = 2dK(x, x)/dx$.

[The function $\theta(x)$ is the Heaviside step function defined as $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$. Its derivative is the Dirac delta function: $d\theta(x)/dx = \delta(x)$.]

SECTION B

Q4 Consider the pair of equations

$$\begin{aligned} u_x &= -uv_x - e^v \\ u_t &= uv_t - e^{-v} \ . \end{aligned}$$

Show that these relations give a Bäcklund transform between an equation for u and an equation for v, that you should find.

Q5 Hirota's bilinear differential operators $D_x^m D_t^n$ are defined by

$$\left(D_x^m D_t^n(f,g)\right)(x,t) := (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n f(x,t)g(x',t')\big|_{x'=x,\ t'=t} \ .$$

By setting $u = \log f$ in the differential equation

$$u_{xt} = e^{-u} - e^{-2u} \ ,$$

show that this equation can be cast into Hirota's bilinear form as

$$D_x D_t(f,f) = H(f) \ ,$$

where H is a function that you should find. Verify that

$$f = 1 + \exp(ax + bt + c)$$

is a solution, provided that the constants a, b, c satisfy appropriate relations that you should find. What is the velocity of this solution?

Q6 Consider $u\,(x,t)$ that satisfies the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

(a) If $D = d/dx$ and $f\,(x)$ is a general function of x, show that

$$\begin{aligned} [D, f] &= f_x, \\ \left[D^2, f\right] &= f_{xx} + 2f_x D. \end{aligned}$$

(b) Consider operators

$$\begin{aligned} L &= D^2 + u\,(x,t)\,, \\ B &= -4D^3 + \alpha_1\,(x)\,D + \alpha_0\,(x)\,, \end{aligned}$$

Find the functions $\alpha_1\,(x)$ and $\alpha_0\,(x)$ of u and u_x such that the KdV equation can be written in the form

$$L_t + [L, B] = 0.$$

You may use that: $[D^3, f] = f_{xxx} + 3f_{xx}D + 3f_x D^2$ *for any function* $f = f\,(x)$, *together with the formulas in part 6(a).*

Q7 Consider the Schrödinger equation

$$\psi'' + \left(k^2 - V(x)\right)\psi = 0,$$

with potential

$$V(x) = \lambda_1 \delta(x+a) - \lambda_2 \delta(x-a),$$

where $\lambda_1, \lambda_2 > 0$.

(a) By imposing the appropriate matching conditions, derive the equation for bound state solutions $k = i\mu$ with $\mu > 0$ in the form

$$(b + 2\mu)(c - 2\mu) - \lambda_1 \lambda_2 e^{\mu d} = 0,$$

giving the constants b, c and d.

(b) Find the bound state solutions (if any) as:

(i) $a \to 0$

(ii) $a \to \infty$

In each case give the corresponding constraints (if any) on λ_1 and λ_2.

(c) Taking for simplicity $\lambda_1 = \lambda_2$ and $a > 0$, is it possible to conclude that there always exists a bound state?

SECTION C

Q8 A field $u(x,t)$ with $x, t \in \mathbb{R}$ is described by the Lagrangian density $\mathcal{L}_L(u, u_x, u_t)$ if $x < 0$, the Lagrangian density $\mathcal{L}_R(u, u_x, u_t)$ if $x > 0$, and the Lagrangian $I(u, u_t)|_{x=0}$ at the interface $x = 0$ between the two regions. The field $u(x,t)$ is continuous at $x = 0$. Use the variational principle (or principle of stationary action) to find the equations obeyed by the field if $x < 0$, $x > 0$ and $x = 0$ respectively. Under which conditions is the equation of motion at the interface $x = 0$ independent of the bulk Lagrangian densities at $x \neq 0$?