Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH4151-WE01

Title: Topics in Algebra and Geometry IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

Q1 (a) Carefully state the valence formula, explaining all terms.

(b) Let $f \in S_{18}(SL_2(\mathbb{Z}))$ be a non-zero cusp form of weight 18 for $SL_2(\mathbb{Z})$. Give all the zeros of f in the upper half plane. Give an example for f.

Q2 Let $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 10 \end{smallmatrix}\right)$, a positive definite integral matrix with associated quadratic form $Q(\mathbf{x}) = \frac{1}{2}{}^t\mathbf{x}A\mathbf{x}$.

(a) Define the associated theta series $\theta(\tau, A)$ and compute its q-expansion up to index 5.

(b) Use the theta transformation formula $\theta(-1/\tau, A) = \frac{-i\tau}{\sqrt{\det A}}\theta(\tau, A^{-1})$ to show

$$\theta\left(\frac{-1}{19\tau}, A\right) = -i\sqrt{19}\tau\theta(\tau, A).$$

Q3 Recall that the arithmetic functions Λ and σ are defined by

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k, \\ 0 & \text{otherwise,} \end{cases}$$
$$\sigma(n) = \sum_{d|n} d.$$

(a) Define the Dirichlet convolution $f * g$ of two arithmetic functions f and g.

(b) Show that the Dirichlet convolution of two multiplicative arithmetic functions is multiplicative.

(c) Using the previous part or otherwise, show that σ is multiplicative and further prove the formula

$$\sigma(n) = \prod_{p^\nu \| n} \frac{p^{\nu+1} - 1}{p - 1}.$$

Here for a prime p, $p^\nu \| n$ means that $p^\nu | n$ but $p^{\nu+1}$ does not divide n.

Q4 Recall that the Gamma function is defined by

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx.$$

(a) Without proof give a complete list of zeroes and poles of $\Gamma(s)$ along with their residues.

(b) Using the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

determine (with proof) all zeroes of $\zeta(s)$ with $Re(s) < 0$.

(c) Show that

$$\zeta(0) = -\frac{1}{2}.$$

You may use here the reflection formula $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$ and that $\zeta(s)$ has a simple pole at 1 with residue 1.

Q5 (a) Let $f(\tau) = 1 + \sum_{n=1}^{\infty} a_n q^n \in M_k(SL_2(\mathbb{Z}))$ with $a_n \in \mathbb{Z}$ for all n.
Let ℓ be the numerator of the k-th Bernoulli number $B_k = \ell/b$ with $gcd(\ell, b) = 1$. Show if $gcd(2k, \ell) = 1$ then there exists a cusp form of weight k for $SL_2(\mathbb{Z})$ with integral Fourier coefficients $b_n \in \mathbb{Z}$ such that

$$b_n \equiv \sigma_{k-1}(n) \pmod{\ell}$$

[You may use the q-expansion of the Eisenstein series $E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$.]

(b) Show using the theory of modular forms that for $k = 14$, the numerator ℓ of B_{14} must divide 28. To which other weights k does your argument extend?

(c) Show that for every even $k \geq 4$ a form $f(\tau) = 1 + \sum_{n=1}^{\infty} a_n q^n \in M_k(SL_2(\mathbb{Z}))$ with $a_n \in \mathbb{Z}$ for all n as in part (a) exists. (This part is independent of the other parts of the question.)

Q6 Consider $M_2(\Gamma_0(19))$, the space of the modular forms of weight 2 for the Hecke subgroup $\Gamma_0(19)$. You may use $\dim M_2(\Gamma_0(19)) = 2$ and that the q-expansion of one element f in the space is given by

$$f(\tau) = 1 + 4q + 4q^2 + 4q^4 + 16q^5 + 16q^6 + O(q^7).$$

Also recall that the action of the Hecke operator T_p of prime index p for $p \neq 19$ on $M_2(\Gamma_0(19))$ is given by "$b_n = a_{pn} + p a_{n/p}$".

(a) Compute the q-expansion of $g := T_2 f$ up to index 3. Show that f and g form a basis of $M_2(\Gamma_0(19))$ and write down the matrix for the T_2 action on $M_2(\Gamma_0(19))$ with respect to this basis.

(b) Find eigenvectors for T_2 and use this to find a normalized eigenbasis h_1 and h_2 of $M_2(\Gamma_0(19))$ for all Hecke operators. Justify your reasoning.

(c) Compute $T_5 f$ and with that also compute $T_5 g$ with $g = T_2 f$ as above. What are the eigenvalues of T_5? Use this to give the Fourier coefficients of index 5 for both h_1 and h_2. Justify your reasoning.

Q7 (a) Show that if $t \in \mathbb{R}$ and $\sigma \geq 1 + 1/(\log|t| + 2)$ then

$$|\zeta(\sigma + it)| \asymp \left| \prod_{p \leq |t|} \left(1 - \frac{1}{p^{\sigma+it}}\right) \right|.$$

Here recall that the notation $A \asymp B$ means that $A \ll B \ll A$. Here, you may use the estimate $|\log(1 - x) + x| \leq |x|^2$ valid for all $|x| < 1/2$.

(b) Show that for any $\sigma > 1$ and any $t \in \mathbb{R}$,

$$\log \zeta(\sigma + it) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma \log(n)} (\cos(\log(n)t) - i \sin(\log(n)t)).$$

Here $\Lambda(n)$ denotes the von Mangoldt function.

(c) Prove that for any $\theta \in \mathbb{R}$,

$$3 + 4\cos(\theta) + \cos(2\theta) \geq 0.$$

(d) Use the previous parts to show that for any $\sigma > 1$ and any $t \in \mathbb{R}$,

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it))| \geq 1.$$

(e) Deduce that $\zeta(1 + it) \neq 0$ for any $t \in \mathbb{R}$. You may use here that ζ only has one simple pole at 1.

Q8 Let q be a positive integer and let ϕ denote the Euler totient function.

Define the Hurwitz zeta function $\zeta(s, \alpha)$ for $Re(s) > 1$ and $\alpha \in (0, 1]$ by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s}.$$

(a) Define what it means for a function $\chi : \mathbb{Z} \to \mathbb{C}$ to be a Dirichlet character modulo q.

(b) Let χ be a Dirichlet character modulo q. Prove that

$$\sum_{1 \leq a \leq q} \chi(a) = \begin{cases} \phi(q) & \text{if } \chi = \chi_0 \text{ is the trivial character,} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Let $L(s, \chi)$ denote the Dirichlet L-function associated to a character χ modulo q. Write down the Euler product for $L(s, \chi)$ for $Re(s) > 1$ and use it to show that $L(s, \chi_0)$ has a simple pole at $s = 1$ with residue $\phi(q)/q$, where χ_0 is the trivial character modulo q. You may assume that ζ has a simple pole with residue 1 at $s = 1$.

(d) Show that for $Re(s) > 1$ and χ a Dirichlet character modulo q, one may write

$$L(\chi, s) = q^{-s} \sum_{a=1}^{q} \chi(a) \zeta\left(s, \frac{a}{q}\right).$$

(e) You are told that $\zeta(s, a/q) - \zeta(s)$ analytically continues to an entire function. Use this information to show that for all *non-trivial* characters χ, $L(s, \chi)$ analytically continues to an entire function on $\mathbb{C}$.