

## EXAMINATION PAPER

Examination Session: May/June

Title:

Year: 2023

Exam Code:

MATH41520-WE01

Topics in Algebra and Geometry V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:

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- **Q1** Let  $f \in S_{18}(SL_2(\mathbb{Z}))$ , a non-zero cusp form of weight 18 for  $SL_2(\mathbb{Z})$ . Use the valence formula to give all the zeros of f in the upper half plane. Give an example for f.
- **Q2** Let  $A = \begin{pmatrix} 2 & 1 \\ 1 & 10 \end{pmatrix}$ , a positive definite integral matrix with associated quadratic form  $Q(\mathbf{x}) = \frac{1}{2}{}^t \mathbf{x} A \mathbf{x}$ .
  - (a) Define the associated theta series  $\theta(\tau, A)$  and compute its *q*-expansion up to index 5.
  - (b) Use the theta transformation formula to show

$$\theta(-1/19\tau, A) = -i\sqrt{19}\tau\theta(\tau, A).$$

**Q3** Recall that the arithmetic functions  $\Lambda$ ,  $\sigma$  are defined by

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k, \\ 0 & \text{otherwise,} \end{cases}$$
$$\sigma(n) = \sum_{d|n} d.$$

(a) Show that  $\sigma$  is multiplicative and further prove the formula

$$\sigma(n) = \prod_{p^{\nu} \parallel n} \frac{p^{\nu+1}-1}{p-1}.$$

Here for a prime p,  $p^{\nu} || n$  means that  $p^{\nu} | n$  but  $p^{\nu+1}$  does not divide n.

(b) Show that for any  $x \ge 1$ 

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \sum_{p \le x} \frac{\log(p)}{p} + O(1).$$

- **Q4** (a) Determine (with proof) all zeroes of  $\zeta(s)$  with Re(s) < 0 along with their residues.
  - (b) Show that

$$\zeta(0)=-\frac{1}{2}.$$

You may use the functional equation and known properties of the  $\Gamma$  function.



- **Q5** (a) Let  $f(\tau) = 1 + \sum_{n=1}^{\infty} a_n q^n \in M_k(SL_2(\mathbb{Z}))$  with  $a_n \in \mathbb{Z}$  for all n. Let  $\ell$  be the numerator of the k-th Bernoulli number  $B_k = \ell/b$  with  $gcd(\ell, b) = 1$ . Show if  $gcd(2k, \ell) = 1$  then there exists a cusp form of weight k for  $SL_2(\mathbb{Z})$  with integral Fourier coefficients  $b_n$  such that  $b_n \equiv \sigma_{k-1}(n) \pmod{\ell}$ . [You may use the q-expansion of the Eisenstein series  $E_k(\tau)$ ].
  - (b) Show using the theory of modular forms that for k = 14, the numerator  $\ell$  of  $B_{14}$  must divide 28. To which other weights *k* does your argument extend?
  - (c) Show that for every even  $k \ge 4$  a form  $f(\tau) = 1 + \sum_{n=1}^{\infty} a_n q^n \in M_k(SL_2(\mathbb{Z}))$  with  $a_n \in \mathbb{Z}$  for all *n* as in part (a) exists. (This part is independent of the other parts of the question.)
- **Q6** Consider  $M_2(\Gamma_0(19))$ , the space of the modular forms of weight 2 for the Hecke subgroup  $\Gamma_0(19)$ . You may use dim  $M_2(\Gamma_0(19)) = 2$  and that the *q*-expansion of one element *f* in the space is given by

$$f(\tau) = 1 + 4q + 4q^2 + 4q^4 + 6q^5 + 16q^6 + O(q^7).$$

Also recall that the action of the Hecke operator  $T_p$  of prime index p for  $p \neq 19$  on  $M_2(\Gamma_0(19))$  is given by " $b_n = a_{pn} + pa_{n/p}$ ".

- (a) Compute the *q*-expansion of  $g := T_2 f$  up to index 3. Show that *f* and *g* form a basis of  $M_2(\Gamma_0(19))$  and write down the matrix for the  $T_2$  action on  $M_2(\Gamma_0(19))$  with respect to this basis.
- (b) Find eigenvectors for  $T_2$  and use this to find a normalized eigenbasis  $h_1$  and  $h_2$  of  $M_2(\Gamma_0(19))$  for all Hecke operators. Justify your reasoning.
- (c) Compute  $T_5 f$  and with that  $T_5 g$ . What are the eigenvalues of  $T_5$ ? Use this to give the Fourier coefficients of index 5 for both  $h_1$  and  $h_2$ . Justify your reasoning.
- **Q7** (a) Show that if  $t \in \mathbb{R}$  and  $\sigma \ge 1 + 1/(\log |t| + 2)$  then

$$|\zeta(\sigma+it)| \asymp \left|\prod_{p\leq |t|} \left(1-\frac{1}{p^{\sigma+it}}\right)\right|.$$

Here recall that the notation  $A \simeq B$  means that  $A \ll B \ll A$ . Here, you may use the estimate  $|\log(1 - x) + x| \le |x|^2$  valid for all |x| < 1/2.

(b) Show that for any  $\sigma > 1$  and any  $t \in \mathbb{R}$ ,

$$\log \zeta(\sigma + it) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma} \log(n)} (\cos(\log(n)t) - i \sin(\log(n)t)).$$

Here  $\Lambda(n)$  denotes the von Mangoldt function.

(c) Prove that for any  $\theta \in \mathbb{R}$ ,

$$3 + 4\cos(\theta) + \cos(2\theta) \ge 0.$$

(d) Use the previous parts to show that for any  $\sigma > 1$  and any  $t \in \mathbb{R}$ ,

$$|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it))|\geq 1.$$



- (e) Deduce that  $\zeta(1 + it) \neq 0$  for any  $t \in \mathbb{R}$ . You may use here that  $\zeta$  only has one simple pole at 1.
- **Q8** Let q be a positive integer and let  $\phi$  denote the Euler totient function.

Define  $\zeta(s, \alpha)$  for Re(s) > 1 and  $\alpha \in (0, 1]$  by

$$\zeta(s,\alpha)=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}.$$

(a) Let  $\chi$  be a Dirichlet character modulo q. Prove that

$$\sum_{1 \le a \le q} \chi(a) = \begin{cases} \phi(q) & \text{if } \chi = \chi_0 \text{ is the trivial character,} \\ 0 & \text{otherwise.} \end{cases}$$

- (b) Let  $L(s, \chi)$  denote the Dirichlet *L*-function associated to a character  $\chi$  modulo q. Write down the Euler product for  $L(s, \chi)$  for Re(s) > 1 and use it to show that  $L(s, \chi_0)$  has a simple pole at s = 1 with residue  $\phi(q)/q$ , where  $\chi_0$  is the trivial character modulo q. You may assume that  $\zeta$  has a simple pole with residue 1 at s = 1.
- (c) Show that for Re(s) > 1 and  $\chi$  a Dirichlet character modulo q, one may write

$$L(\chi, s) = q^{-s} \sum_{a=1}^{q} \chi(a) \zeta\left(s, \frac{a}{q}\right)$$

(d) You are told that ζ (s, a/q) – ζ(s) analytically continues to an entire function. Use this information to show that for all *non-trivial* characters χ, L(s, χ) analytically continues to an entire function on C.