



EXAMINATION PAPER

Examination Session: May/June	Year: 2023	Exam Code: MATH4161-WE01
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Title: Algebraic Topology IV
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Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.</p> <p>Students must use the mathematics specific answer book.</p>
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Revision:	
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The following conventions hold in this paper:

The n -dimensional sphere with the Euclidean topology is denoted by S^n .

The closed n -dimensional unit disc with the Euclidean topology is denoted by D^n .

SECTION A

Q1 Let X be a non-empty topological space.

- (a) State the definition of the cone of X , denoted CX .
- (b) Show that CX is contractible.
- (c) State the definition of the suspension of X , denoted SX .
- (d) Show that if X is contractible, then SX is contractible.

Q2 Let $H_k(X)$ denote the k -th singular homology group of a topological space X with coefficients in \mathbb{Z} .

- (a) State the definition of the degree of a map $f: S^n \rightarrow S^n$ with $n \geq 1$.
- (b) Construct a CW-complex X with

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/3 & k = 2, \\ \mathbb{Z}/9\mathbb{Z} & k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Justify your construction. You can assume the existence of maps of specific degree.

Q3 In this exercise we denote by \mathbb{RP}^3 the 3-dimensional real projective space.

- (a) Describe \mathbb{RP}^3 as a CW-complex, deduce the complex underlying its CW-homology, and compute its homology.
- (b) Provide a compact, 4-dimensional manifold with non-empty boundary W which is homotopy-equivalent to $Y = \mathbb{RP}^3$. Describe the homotopy-equivalence explicitly.
- (c) Is there a closed, connected 4-dimensional oriented manifold X which is homotopy-equivalent to \mathbb{RP}^3 ?

Q4 Let M be an n -dimensional connected, oriented manifold with non-empty connected boundary ∂M , and let $n \geq 2$. The boundary ∂M is itself an orientable manifold (you do not need to prove this).

- (a) State the Poincaré-Lefschetz duality theorem which holds in this situation.
- (b) Compute $H_n(M)$.
- (c) In this exercise you can use that the relative cohomology group $H^1(M, \partial M; \mathbb{Z})$ has no torsion, without proving it. Conclude that in the long exact homology sequence of the pair $(M, \partial M)$ the connecting homomorphism $H_n(M, \partial M) \rightarrow H_{n-1}(\partial M)$ is an isomorphism.

SECTION B

- Q5** (a) Let (C_*, ∂_*) be a chain complex over \mathbb{Z} . Assume that there exist homomorphisms $H_n: C_n \rightarrow C_{n+1}$ with $H_{n-1} \circ \partial_n + \partial_{n+1} \circ H_n = \text{id}_{C_n}$ for all $n \in \mathbb{Z}$. Show that $H_n(C_*) = 0$ for all $n \in \mathbb{Z}$.
- (b) Consider the chain complex (C_*, ∂_*) where $C_n = 0$ for $n < 0$ or $n > 4$, and the remaining chain groups fit in a sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & -1 \end{pmatrix}} \mathbb{Z} \longrightarrow 0$$

Show that $H_0: C_0 \rightarrow C_1$ given by

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}$$

can be extended to homomorphisms $H_n: C_n \rightarrow C_{n+1}$ with

$$H_{n-1} \circ \partial_n + \partial_{n+1} \circ H_n = \text{id}_{C_n}$$

for all $n \in \mathbb{Z}$.

- (c) Consider the chain complex (D_*, ∂_*) given by $D_n = \mathbb{Z}/4\mathbb{Z}$ and $\partial_n: D_n \rightarrow D_{n-1}$ given by multiplication with $2 \in \mathbb{Z}/4\mathbb{Z}$ for every $n \in \mathbb{Z}$. Calculate $H_n(D_*)$ for every $n \in \mathbb{Z}$.
- (d) Decide whether there exists $H_n: D_n \rightarrow D_{n+1}$ with $H_{n-1} \circ \partial_n + \partial_{n+1} \circ H_n = \text{id}_{D_n}$ for all $n \in \mathbb{Z}$. Justify your statement.

Q6 In this question $H_k(X)$ is the k -th singular homology group of a topological space X with coefficients in \mathbb{Z} . Let $f: S^1 \rightarrow S^1$ be a map of degree 2. For n a positive integer let

$$T_n = \left(\prod_{i=0}^n S^1 \times [i, i+1] \right) / \sim,$$

where $(z, i+1) \in S^1 \times [i, i+1]$ is identified by \sim with $(f(z), i+1) \in S^1 \times [i+1, i+2]$ for all $i = 0, \dots, n-1$.

- Let $i_n: T_n \rightarrow T_{n+1}$ be the function given by $i_n([z, t]) = [z, t]$ for $[z, t] \in T_n$. Show that i_n is an embedding (that is, an injective map which is a homeomorphism onto its image) of T_n into T_{n+1} .
- Calculate $H_k(T_n)$ for all $k \geq 0$ and $n \geq 1$, and determine the homomorphism $i_{n*}: H_k(T_n) \rightarrow H_k(T_{n+1})$ for all $n, k \geq 1$.
- We now treat $T_n \subset T_{n+1}$ using i_n from the previous question. Let

$$T = \bigcup_{n=1}^{\infty} T_n$$

and topologise T as follows: A subset $A \subset T$ is closed if and only if $A \cap T_n$ is closed in T_n for all $n \geq 1$. Show that the set-theoretic inclusion $j_n: T_n \rightarrow T$ is an embedding for all $n \geq 1$.

- Show that $j_{n*}: H_1(T_n) \rightarrow H_1(T)$ is injective, and $H_1(T)$ does not contain torsion elements. Furthermore, show that for each $x \in H_1(T)$ there exists $y \in H_1(T)$ with $2y = x$.

Hint: You may use the following result from the lectures without proof: Let X be a topological space and $(U_n)_{n \in \mathbb{N}}$ open sets with $U_n \subset U_{n+1}$ and $X = \bigcup_{n=1}^{\infty} U_n$. Then

- For each $\alpha \in H_k(X)$ there is $n \in \mathbb{N}$ with $\alpha \in \text{im}(j_{n*}: H_k(U_n) \rightarrow H_k(X))$.
- If $\alpha_n \in H_k(U_n)$ satisfies $j_{n*}\alpha_n = 0 \in H_k(X)$, then there is $m > n$ with $j_{n,m*}\alpha_n = 0 \in H_k(U_m)$, where $j_{n,m}: U_n \rightarrow U_m$ is inclusion.

Q7 Consider the sequence of abelian groups

$$0 \rightarrow C_2 := \mathbb{Z}/6 \xrightarrow{\partial_2 := \times 2} C_1 := \mathbb{Z}/6 \xrightarrow{\partial_1 := \times 3} C_0 := \mathbb{Z}/6 \rightarrow 0,$$

where $\times 2: \mathbb{Z}/6 \rightarrow \mathbb{Z}/6$ denotes the map induced from multiplication with 2 on \mathbb{Z} , and likewise $\times 3: \mathbb{Z}/6 \rightarrow \mathbb{Z}/6$ denotes the map induced from multiplication with 3 on \mathbb{Z} .

- (a) Show that this sequence is a complex of abelian groups (C_*, ∂_*) .
- (b) Determine its homology groups $H_*(C_*)$.
- (c) Determine the cohomology groups $H^*(C^*; \mathbb{Z}/6)$ with coefficients in $\mathbb{Z}/6$.
- (d) State the groups $\text{Ext}^1(\mathbb{Z}/2, \mathbb{Z}/6)$ and $\text{Ext}^1(\mathbb{Z}/3, \mathbb{Z}/6)$. (It is not required to derive these using a resolution.)
- (e) Explain why the homology groups $H_*(C_*)$ and $H^*(C^*; \mathbb{Z}/6)$ cannot arise as singular homology $H_*(X)$ and singular cohomology $H^*(X; \mathbb{Z}/6)$ for a finite CW-complex X .

Q8 The aim of this problem is to conclude that there is no continuous map

$$f: S^2 \times S^2 \rightarrow \mathbb{CP}^2$$

of degree ± 1 . In this exercise M will always denote a n -dimensional closed connected oriented manifold for some integer $n \geq 2$, and we denote by $[M] \in H_n(M; \mathbb{Z})$ its fundamental class.

- (a) Show that $H^n(M; \mathbb{Z})$ is isomorphic to \mathbb{Z} .
- (b) Show that the evaluation map $\text{ev}: H^n(M; \mathbb{Z}) \rightarrow \text{Hom}(H_n(M), \mathbb{Z})$ is an isomorphism.
- (c) Conclude that there is a unique element $[M]^* \in H^n(M; \mathbb{Z})$ such that one has $\langle [M]^*, [M] \rangle = 1$. This is called the dual fundamental class.
- (d) Suppose $f: M \rightarrow N$ is a map of degree $\deg(f) = d \in \mathbb{Z}$, where N is a n -dimensional closed connected oriented manifold. Show that one has

$$f^*([N]^*) = d \cdot [M]^*.$$

- (e) Show that there is no map $g: S^4 \rightarrow \mathbb{CP}^2$ of non-zero degree d . (Hint: Use the fact that g^* is a ring homomorphism.)
- (f) Show that for any element $a \in H^2(S^2 \times S^2; \mathbb{Z})$ the expression

$$\langle a \smile a, [S^2 \times S^2] \rangle$$

is an even integer.

- (g) Conclude that there is no map $f: S^2 \times S^2 \rightarrow \mathbb{CP}^2$ of odd degree (and in particular not of degree ± 1).