

## **EXAMINATION PAPER**

Examination Session: May/June

2023

Year:

Exam Code:

MATH41620-WE01

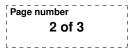
Title:

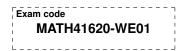
Number Theory V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:





## SECTION A

- **Q1** (a) Let  $K = \mathbb{Q}(\sqrt{7})$ . Compute the discriminant  $\Delta_K(2, 3 + \sqrt{7})$ .
  - (b) Let  $K = \mathbb{Q}(\theta)$ , where  $\theta$  is a root of the polynomial  $7x^2 3$ . Determine the ring of integers  $\mathcal{O}_K$  and give a generator for its group of units  $\mathcal{O}_K^{\times}$ .
- **Q2** (a) Let K be a number field and I and J two ideals of  $\mathcal{O}_K$  such that  $I \subseteq J$ . Recall that every ideal of  $\mathcal{O}_K$  is a full lattice in K and hence has a discriminant. Prove that if  $\Delta_K(I) = \Delta_K(J)$ , then I = J.
  - (b) Find all the ideals of  $\mathbb{Z}[\sqrt{-21}]$  that contain the element 10 and have norm 10.
- **Q3** Let  $K = \mathbb{Q}(\sqrt{-5})$  and consider the following ideals of  $\mathcal{O}_K$ :

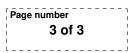
$$p = (2, 1 + \sqrt{-5}),$$
  

$$q = (3, 1 + \sqrt{-5}),$$
  

$$r = (3, 1 - \sqrt{-5}).$$

- (a) Using Kummer–Dedekind or otherwise, show that  $\mathfrak{p},\mathfrak{q},\mathfrak{r}$  are prime ideals and find their norms.
- (b) Show that  $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$  are not principal ideals.
- (c) Show that in the class group  $Cl(\mathcal{O}_K)$ , we have the relations

$$[\mathfrak{p}]^2 = e, \qquad [\mathfrak{p}][\mathfrak{q}] = e.$$





## SECTION B

**Q4** (a) Let R be any integral domain and suppose that  $x \in R$  has a factorisation

$$x = up_1 \cdots p_n,$$

where u is a unit and the  $p_i$  are *prime elements*. Show that the above factorisation is the unique factorisation of x into a product of *irreducible* elements, up to ordering of the factors and multiplication by units. (Hint: Let  $x = vq_1 \cdots q_m$ be a factorisation where v is a unit and the  $q_i$  are irreducible.)

- (b) Let r be an element of an integral domain R and consider the principal ideal (r) of R. Show that if (r) is a prime ideal, then r is a prime element.
- (c) It is known that the ring  $R = \mathbb{Z}[\sqrt{-5}]$  is not a UFD (i.e., does not have unique factorisation into irreducibles). Show that, nevertheless, the element  $5 \in R$  does factor uniquely into a product of irreducibles (up to units and ordering of factors).
- **Q5** Let  $\theta \in \mathbb{C}$  be a root of the polynomial  $x^3 3$  and let  $K = \mathbb{Q}(\theta)$ .
  - (a) Compute  $\Delta_K(\mathbb{Z}[\theta])$ .
  - (b) Show that  $\mathcal{O}_K = \mathbb{Z}[\theta]$ .
- **Q6** Let  $K = \mathbb{Q}(\sqrt{d})$ , where  $d \in \mathbb{Z}$  is squarefree and  $d \not\equiv 1 \pmod{4}$ . Let  $a, b, c \in \mathbb{Z}$  with a and c non-zero and such that

$$c \mid a, c \mid b$$
 and  $ac \mid (b^2 - dc^2)$ .

- (a) Show that there exist  $n, r \in \mathbb{Z}$  such that bn dc = ar.
- (b) Prove that

 $\{a, b + c\sqrt{d}\}$ 

is a generating basis (i.e., a  $\mathbb{Z}$ -basis) for the ideal  $(a, b + c\sqrt{d})$  of  $\mathcal{O}_K$ .

- (c) Compute the norm of the ideal  $(6, 5 \sqrt{7})$ .
- **Q7** Determine the class group of  $K = \mathbb{Q}(\sqrt{-57})$ . You may use the Minkowski bound, given by  $B_K = \left(\frac{4}{\pi}\right)^t \frac{n!}{n^n} \sqrt{|\Delta_K|}$ .

## SECTION C

- **Q8** (a) Write  $-\frac{6}{5}$  as a 3-adic expansion, that is, in the form  $a_0a_1...$ , where  $-\frac{6}{5} = \sum_{i=0}^{\infty} a_i 3^i$ , for  $a_i \in \{0, 1, 2\}$ .
  - (b) Write  $\frac{6}{5}$  as a 3-adic expansion.