EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH4171-WE01

Title:

Riemannian Geometry IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Prove or disprove the following assertions:

1.1 *The set*

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 - 3z^2 = 1\}$$

is a smooth 2*-dimensional submanifold of* $\mathbb{R}^3$*.*

1.2 *The subset of* $\mathbb{R}^3$ *given by*

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid y = z = 0 \text{ and } x \geq 1\}$$

is a smooth submanifold of $\mathbb{R}^3$*.*

Q2 **2.1** State the definition of a differentiable vector field on a differentiable manifold.

2.2 Let X, Y be two differentiable vector fields on $\mathbb{R}^2$ defined by

$$X(x, y) = \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

$$Y(x, y) = (2xy - x^2)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Compute the Lie bracket $[X, Y]$.

2.3 Show that $[X, Y]$ restricts to a vector field on the one-dimensional submanifold $P = \{(x, y) \in \mathbb{R}^2 \mid x = y^2\}$ of $\mathbb{R}^2$.

Q3 **3.1** Let V be a finite dimensional real vector space. Give the definition of a tensor of type $\binom{k}{l}$ of the vector space V.

3.2 Let M be a differentiable manifold with two different Riemannian metrics g and $\widetilde{g}$ and corresponding Levi-Civita connections ∇ and $\widetilde{\nabla}$, respectively. Let X be a fixed non-zero vector field on M. For an arbitrary vector field Y on M, let

$$\begin{aligned} T_1(Y) &:= \nabla_X Y - \nabla_Y X, \\ T_2(Y) &:= \nabla_X Y - \widetilde{\nabla}_X Y. \end{aligned}$$

Using the Tensor Characterisation Lemma or otherwise, prove or disprove for each of the maps T_j that they are tensor fields. You can use without proof that, if $X(p) \neq 0$, there exists a function $f \in C^\infty(M)$ with $f(p) = 0$ and $X(f)(p) \neq 0$.

Q4 Let (M, g) be a Riemannian manifold and $c : [0, a] \to M$ be a unit speed geodesic.

4.1 Give the definition of a Jacobi field along c.

4.2 Let J be a Jacobi field along c. Show that the function

$$f(t) = g(J(t), c'(t))$$

is a linear function and can be expressed as

$$f(t) = g(J(0), c'(0)) + tg(J'(0), c'(0)),$$

where $J' = \frac{D}{dt}J$ denotes the covariant derivative of J along c.

4.3 Show that if a Jacobi field J along c satisfies $J(0) = 0$ and $J'(0) \perp c'(0)$, then it is perpendicular to c at every point $t \in [0, a]$.

SECTION B

Q5 **5.1** Consider the smooth manifolds

$$M = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0\}$$

and

$$N = \{(u, v) \in \mathbb{R}^2 \mid u, v > 0\}$$

with global charts given by the identity map. Let $f \colon M \to N$ be the map given by $f(x, y) = (xy, y/x)$. Show that f is a diffeomorphism and, given the smooth vector field

$$X(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

on M, compute the expression of the vector field

$$Y(u, v) = Df_{(x,y)}X(x, y)$$

on N in terms of the basis determined by the coordinate tangent vectors $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$.

5.2 Let $\mathrm{SO}(n)$ be the special orthogonal group in dimension $n \geq 2$. Show that, for any $C \in \mathrm{SO}(n)$, the map $f_C \colon \mathrm{SO}(n) \to \mathrm{SO}(n)$ given by $f_C(A) = CAC^{-1}$ is a Lie group isomorphism.

5.3 Let G be a Lie group, let e be its identity element, and let $\mu \colon G \times G \to G$ be the product map, given by $\mu(g, h) = gh$. Using the identification $T_{(e,e)}(G \times G) = T_eG \oplus T_eG$, show that the differential $D\mu_{(e,e)} \colon T_eG \oplus T_eG \to T_eG$ is given by

$$D\mu_{(e,e)}(u, v) = u + v$$

for all $(u, v) \in T_eG \oplus T_eG$.

Q6 **6.1** Give the definition of an isometry between Riemannian manifolds.

6.2 Prove or disprove the following assertion: *The unit two-dimensional round sphere $S^2(1)$ is isometric to the unit three-dimensional round sphere $S^3(1)$.*

6.3 Let (M_1, g_1), (N_1, h_1) and (M_2, g_2), (N_2, h_2) be two pairs of isometric Riemannian manifolds, i.e., (M_1, g_1) is isometric to (N_1, h_1) and (M_2, g_2) is isometric to (N_2, h_2). Show that $M_1 \times M_2$ and $N_1 \times N_2$, equipped with the product Riemannian metrics, are isometric Riemannian manifolds. You may use, without proof, results on product manifolds and product Riemannian metrics from lectures.

Q7 For Questions 7.2 and 7.3, let $U \subset \mathbb{R}^2$ be a small open set around the origin with Riemannian metric given by

$$g_{ij} = \frac{\delta_{ij}}{F^2} \tag{1}$$

with a smooth nowhere zero function $F \in C^\infty(U)$.

7.1 Give the formula for the computation of Christoffel symbols from the metric g_{ij} for an arbitrary n-dimensional Riemannian manifold (M, g).

7.2 For the 2-dimensional manifold (U, g) with metric (1) above, show that the sectional curvature K is given by

$$K = F(F_{11} + F_{22}) - (F_1)^2 - (F_2)^2,$$

with $F_1 = \frac{\partial F}{\partial x}$, $F_2 = \frac{\partial F}{\partial y}$, $F_{11} = \frac{\partial^2 F}{\partial x^2}$ and $F_{22} = \frac{\partial^2 F}{\partial y^2}$. Note that the sectional curvature is usually a function on 2-planes of the tangent bundle, but since the tangent spaces T_pU are 2-dimensional, the sectional curvature can be viewed as a function of $p \in U$, that is $K(p) := K(T_pU)$.

7.3 Compute the sectional curvature of (U, g) in the case $F(x, y) = 1 + x^2 + y^2$.

Q8 **8.1** Let (M, g) be a Riemannian manifold with constant sectional curvature $K(\Sigma) = C$ for all 2-planes $\Sigma \subset TM$. Show that we then have, for every unit speed geodesic $c : [0, a] \to M$,

$$R(v, c'(t))c'(t) = Cv \quad \text{for all } t \in [0, a] \text{ and } v \perp c'(t).$$

Hint: Check that for unit vectors $v, w \in T_{c(t)}M$ with $v \perp w$ and $v, w \perp c'(t)$, we have

$$\langle R(v, c'(t))c'(t), v\rangle = \langle R(w, c'(t))c'(t), w\rangle = C$$

and

$$\langle R(v + w, c'(t))c'(t), v + w\rangle = 2C.$$

Conclude from this that $\langle R(v, c'(t))c'(t), w\rangle = 0$ and use this to prove the statement.

8.2 Use Question 8.1 to show that we have the following on manifolds (M, g) with constant sectional curvature $C < 0$: If $c : [0, a] \to M$ is a unit speed geodesic and $X : [0, a] \to TM$ a parallel vector field along c perpendicular to c, then $J : [0, a] \to TM$ with $J(t) = e^{t\sqrt{|C|}}X(t)$ is a Jacobi field along c.

8.3 Let (M, g) be a Riemannian manifolds and $\varphi : (-\epsilon, \epsilon) \times M \to M$ be a smooth map such that, for all $s \in (-\epsilon, \epsilon)$, the map $\varphi_s : M \to M$, defined by $\varphi_s(p) = \varphi(s, p)$, is a global isometry. Assume further that φ_0 is the identity map of M, that is $\varphi_0(p) = p$ for all $p \in M$. Let X be the vector field defined by

$$X(p) = \frac{d}{ds}|_{s=0}\, \varphi_s(p).$$

Let $c : [0, a] \to M$ be a geodesic. Show that the restriction $t \mapsto X(c(t))$ is a Jacobi field along c.