

EXAMINATION PAPER

Examination Session: May/June

2023

Year:

Exam Code:

MATH41720-WE01

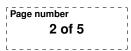
Title:

Partial Differential Equations V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85
		series.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:



SECTION A

Q1 Consider the Cauchy problem for Burgers' equation

$$\begin{cases} \partial_t u(x,t) + \frac{1}{2} \partial_x (u^2(x,t)) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$
(1)

where $u_0 : \mathbb{R} \to \mathbb{R}$ is given.

- **1.1** Let $u_0(x) = \frac{1}{7}x^7$. Show that (1) has a global in time classical solution.
- **1.2** Let $u_0(x) = \sin(x)$. Write down the definition of the critical time t_c (until when we can guarantee the existence of a classical solution to (1)) associated to this initial datum. Show that $t_c \leq 1$.

Q2 In this problem we consider harmonic function on the unit ball in \mathbb{R}^3 , $B_1(0)$.

2.1 Using the fact that the Laplacian in spherical coordinates is given by

$$\Delta \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

show that if $u \in C^2\left(\overline{B_1(0)}\right)$ is radial (i.e. only depends on r in spherical coordinates) and harmonic in $B_1(0)$ then it must be constant.

2.2 Show that there exists no radial solution in $C^2\left(\overline{B_1(0)}\right)$ to the equation

$$\begin{cases} -\Delta u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in B_1(0), \\ u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \partial B_1(0), \end{cases}$$

for $f(x) = x_1^2$.

Q3 Consider the heat-like equation

$$\begin{cases} u_t - u_{xx} + cu = 0, & (x, t) \in \mathbb{R} \times (0, +\infty), \\ u(x, 0) = g(x), & x \in \mathbb{R}, \end{cases}$$
(2)

where $c \in \mathbb{R}$ is a fixed constant and $g \in C_c(\mathbb{R})$.

3.1 Define $v(x,t) = e^{ct}u(x,t)$. Show that v(x,t) solves the heat equation

$$\begin{cases} v_t - v_{xx} = 0, \quad (x,t) \in \mathbb{R} \times (0,+\infty), \\ v(x,0) = g(x), \quad x \in \mathbb{R}. \end{cases}$$

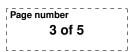
3.2 Show that there exists a solution to (2) that satisfies

$$\sup_{x\in\mathbb{R}}|u(x,t)|\leq e^{-ct}\,\|g\|_{L^{\infty}(\mathbb{R})}$$

You may use the following inequality without proof: For any $f \in L^1(\mathbb{R})$ and $g \in L^{\infty}(\mathbb{R})$ we have that

$$\left| \int_{\mathbb{R}} f(x-y) g(y) \right| \le \|f\|_{L^{1}(\mathbb{R})} \|g\|_{L^{\infty}(\mathbb{R})}, \qquad \forall x \in \mathbb{R}.$$

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SECTION B

Q4 We consider the following conservation law

$$\begin{cases} \partial_t u(x,t) - u(x,t)\partial_x u(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$
(3)

[Notice that this is *not* Burgers' equation.]

- **4.1** Suppose that u_0 is bounded, differentiable with bounded derivative. Give a formula of the critical time t_c , for which we know that (3) has a classical solution on $\mathbb{R} \times (0, t_c)$.
- **4.2** Let $u_0(x) = -\arctan(x)$. Show that in this case (3) has a global in time classical solution.
- **4.3** Let u_0 now be given by

$$u_0(x) = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0. \end{cases}$$

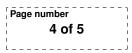
By drawing the characteristics, show that there is instantaneous crossing of characteristics. Find a shock that satisfies the Rankine–Hugoniot condition. Give the expression of the weak solution in this case.

Q5 We aim to solve the following problem by the method of characteristics

$$\begin{cases} \partial_{xx}^2 u - 3\partial_{xy}^2 u + 2\partial_{yy}^2 u = 0, & (x, y) \in \mathbb{R}^2, \\ u(1, y) = g(y), & y \in \mathbb{R}, \\ \partial_x u(1, y) = h(y), & y \in \mathbb{R}, \end{cases}$$
(4)

where $g, h : \mathbb{R} \to \mathbb{R}$ are given smooth functions.

- 5.1 Identify the Cauchy data and the Cauchy curve in the above problem.
- **5.2** Rewrite the PDE in (4) as a system of two linear first order PDEs. [*Hint*: think about the algebraic relation $(a b)(a 2b) = a^2 3ab + 2b^2$, $(a, b \in \mathbb{R})$.]
- 5.3 By solving the two first order PDEs arising from 5.2 using the method of characteristics, find the solution to (4).



Q6 Let Ω be an open bounded set with smooth boundary in \mathbb{R}^n and let u_1 and u_2 be $C^2(\Omega) \cap C(\overline{\Omega})$ solutions to Poisson equation

Exam code

MATH41720-WE01

$$\begin{cases} -\Delta u_i(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u_i(\boldsymbol{x}) = g_i(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega, \end{cases}$$

- i = 1, 2, where $f \in C^1(\overline{\Omega})$ and $g_1, g_2 \in C(\partial \Omega)$.
- **6.1** Show that for any $\boldsymbol{x} \in \overline{\Omega}$

$$u_2({oldsymbol x}) - u_1({oldsymbol x}) \leq \max_{{oldsymbol x} \in \partial \Omega} \left(g_2({oldsymbol x}) - g_1({oldsymbol x})
ight).$$

6.2 Show that

$$\max_{\boldsymbol{x}\in\overline{\Omega}}|u_2(\boldsymbol{x})-u_1(\boldsymbol{x})|\leq \max_{\boldsymbol{x}\in\partial\Omega}|g_2(\boldsymbol{x})-g_1(\boldsymbol{x})|\,.$$

6.3 For $n \in \mathbb{N}$ let $u_n \in C^2(\Omega) \cap C(\overline{\Omega})$ solve the system

$$\begin{cases} -\Delta u_n(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u_n(\boldsymbol{x}) = g_n(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega, \end{cases}$$

and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solve the system

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \partial \Omega. \end{cases}$$

Show that if $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly to g on $\partial\Omega$ then $\{u_n\}_{n\in\mathbb{N}}$ converges uniformly to u on $\overline{\Omega}$.

Recall that we say that a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in C(K) converges uniformly to $f \in C(K)$ if

$$\sup_{x \in K} |f_n(x) - f(x)| \underset{n \to \infty}{\longrightarrow} 0.$$

Q7 Let $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a classical solution to the equation

$$\begin{cases} u_t + ku_{xxxx} = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where k > 0 is a fixed constant and f is a smooth function on \mathbb{R} that belongs to $L^{1}(\mathbb{R}) \cap L^{2}(\mathbb{R})$.

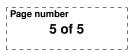
7.1 Show that \hat{u} , the Fourier transform of u in the x-variable, satisfies

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)e^{-k\xi^4 t}.$$

7.2 Using the fact that the Fourier transform preserves the L^2 norm (Plancherel's identity) show that

$$\|u(\cdot,t)\|_{L^{2}(\mathbb{R})}^{2} \leq \frac{\int_{\mathbb{R}} e^{-x^{4}} dx}{\sqrt[4]{2kt}} \|f\|_{L^{2}(\mathbb{R})}^{2}$$

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SECTION C

- **Q8** For $n \in \mathbb{N}$ we define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + 1/n}$.
 - **8.1** Show that as $n \to +\infty$, the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly to f(x) = |x|.
 - **8.2** Show that the sequence $\{f'_n\}_{n\in\mathbb{N}}$ also converges pointwise as $n \to +\infty$. Is this convergence uniform? Justify your answer.
 - **8.3** Show that as $n \to +\infty$, the sequence $\{f_n''\}_{n \in \mathbb{N}}$ converges to $2\delta_0$ in the sense of distributions, where δ_0 is the Dirac mass concentrated at 0.