Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH41920-WE01

Title: Geometry V

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

SECTION A

Q1 Let r be the reflection on the Euclidean plane with respect to the line $x = 0$. Let R be the anticlockwise rotation about the origin $O = (0, 0)$ though the angle $\pi/2$.

(a) What is the type of the transformation $\psi = R \circ r \circ R^{-1}$?

(b) Find the fixed points of the transformation ψ described in part (a).

Q2 (a) Is it true or false that affine transformations act transitively on quadrilaterals in the Euclidean plane? Justify your answer.

(b) A hexagon $ABCDEF$ in the Euclidean plane is symmetric with respect to the diagonal AD. Is it always true that there exists a projective map taking $ABCDEF$ to a regular hexagon? Justify your answer.

Q3 Let $ABCDE$ be a hyperbolic pentagon with $AB = BC = CD = DE = a$ and $\angle ABC = \angle BCD = \angle CDE = \pi/2$.

(a) Let $\gamma = \angle BCA$. Express $\sin\gamma$ in terms of a.

(b) Express the length of AE in terms of a.

SECTION B

Q4 (a) Does there exist a Möbius transformation taking the points $-1, 0, 1+i, 2+i$ to the points $5, 4+3i, -3+4i, -4-3i$ respectively? Justify your answer.

(b) Let γ_1 and γ_2 be two circles with centres O_1 and O_2 respectively on the Euclidean plane. Let f be a Möbius transformation taking γ_1 to γ_2. Is it always true that $f(O_1) = O_2$? Justify your answer.

(c) Consider the four circles of radius 1 centred at the points $1+i, -1+i, -1-i, 1-i$. How many different Möbius transformations take the union of the four circles to itself (not necessarily pointwise)? Justify your answer.

Q5 (a) Show that there exists a regular hyperbolic quadrilateral with all angles equal to $\pi/3$.

(b) Let $ABCD$ be a regular hyperbolic quadrilateral with all angles equal to $\pi/3$ labelled in the clockwise direction. For every $X \in \mathbb{H}^2$ denote by $R_{\pi/3,X}$ a rotation about X through the angle $\pi/3$ (in anti-clockwise direction). Denote $f = R_{\pi/3,D} \circ R_{\pi/3,C} \circ R_{\pi/3,B} \circ R_{\pi/3,A}$. Find the type of the isometry f.

(c) Let M be the midpoint of AD. Find $(R_{\pi/3,B} \circ R_{\pi/3,A})^{2023}(M)$.

Q6 (a) Let $A_1A_2A_3A_4$ be a quadrilateral on the Euclidean plane. Let B_i, $i = 1, \ldots, 4$, be a midpoint of A_iA_{i+1} (where $A_5 = A_1$). Show that $B_1B_2 = B_3B_4$ and $B_2B_3 = B_1B_4$.

(b) Show that the statement of (a) does not hold for a spherical quadrilateral $A_1A_2A_3A_4$.

(c) Consider the spherical quadrilateral $A_1A_2A_3A_4$ and the corresponding quadrilateral $B_1B_2B_3B_4$ constructed as in part (a). Let $S_{B_1B_2B_3B_4}$ and $S_{A_1A_2A_3A_4}$ be the areas of the two quadrilaterals. Is it true that $S_{B_1B_2B_3B_4} = \frac{1}{2}S_{A_1A_2A_3A_4}$? Justify your answer.

Q7 (a) Find the cross-ratio of the following four points in $\mathbb{R}P^2$ (given in homogeneous coordinates):

$$A = (1:0:0), \quad B = (1:1:1), \quad C = (0:1:1), \quad D = (-2:1:1).$$

(b) Let $\triangle A_1A_2A_3$ be a triangle and T be a point on the Euclidean plane. Assume that $T \notin A_iA_j$ for $i, j \in \{1,2,3\}$, $i \neq j$. Let $B_i = A_iT \cap A_jA_k$ for $i = 1,2,3$ and $i, j, k \in \{1,2,3\}$ distinct indices. Let $C_i = A_jA_k \cap B_jB_k$ for $i = 1,2,3$ and $i, j, k \in \{1,2,3\}$ distinct indices. Assuming that all the points listed above are distinct and exist, prove that the points C_1, C_2, C_3 are collinear.

(c) Formulate the statement dual to the one given in part (b).

SECTION C

Q8 (a) Formulate a projective classification of plane conics (you do not need to prove it). For each item in the classification, provide an example. For which of the items above there exists another example which is different from the first example with respect to the metric classification? Justify your answer.

(b) Given a parabola, let A be a point on the parabola, F be a focus of the parabola and H be an orthogonal projection of A to the directrix of the parabola. Let l be a tangent line to the parabola at the point A. Show that l bisects the angle $\angle FAH$.