

# EXAMINATION PAPER

Examination Session: May/June

2023

Year:

Exam Code:

MATH42320-WE01

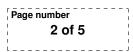
### Title:

## Statistical Mechanics V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.	

Revision:



#### SECTION A

**Q1** Consider a thermodynamic system whose temperature T and pressure p satisfy the following relations:

$$T(E,V) = \frac{\gamma E}{Nk_B}$$
  $p(E,V) = \frac{\gamma E}{V}$ 

Here E is the internal energy, V the volume, and N the number of particles.  $\gamma$  is a constant. You may take the first law of thermodynamics to be dE = TdS - pdV.

- (a) Using the relations above, find an expression for the entropy S that is consistent with the above. Is the answer you found for S unique?
- (b) Your colleague claims that they have found a system whose temperature and pressure satisfy the relations

$$T = \frac{\gamma E}{Nk_B} \qquad p = \frac{\gamma E}{V} - \beta \frac{E^2}{V^2}$$

where  $\beta$  is a non-zero constant. Is this consistent with the laws of thermodynamics? Explain your answer.

**Q2** Consider a classical particle moving in one dimension q with the usual free particle Hamiltonian:

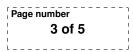
$$H(q,p) = \frac{p^2}{2m} \tag{1}$$

For this problem you may need the Poisson bracket  $\{A, B\}$  between two functions A, B on phase space:  $\{A, B\} = \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q}$ . You may also need Liouville's equation:  $\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0$ .

- (a) Compute the Poisson brackets  $\{H, p\}, \{H, q\}, \text{ and } \{q, p^2\}.$
- (b) We now study probability distributions  $\rho(q, p; t)$  that evolve according to Hamiltonian evolution with Hamiltonian (1). Which of the following probability distributions are time-independent solutions to Liouville's equation? Briefly explain your reasoning:
  - (i)  $\rho_1(q,p) = \mathcal{N}_1 \exp(-\beta_1 H)$
  - (ii)  $\rho_2(q,p) = \mathcal{N}_2 \exp(-\beta_2 p^4)$
  - (iii)  $\rho_3(q,p) = \mathcal{N}_3 \exp(-\beta_3 q^2)$
- (c) Show that a probability distribution of the form

$$\rho(q, p) = f(pt - \alpha q)$$

is a solution to Liouville's equation with Hamiltonian (1) for an appropriate choice of the constant  $\alpha$ , which you should find. Here f(x) is an arbitrary function of one variable.



- **Q3** Consider a system of N particles confined inside a volume V. Moreover, the single particle density of states is given by the function g(E, V, N).
  - (a) Express the partition function  $Z(\beta, V, N)$  in terms of the density of states g(E, V, N).
  - (b) Compute the partition function for the case where,

$$g(E, V, N) = f(V, N) E^n,$$

when E > 0 and zero otherwise. Your final answer should depend on the positive function f(V, N) and the positive integer n.

- (c) For the same system as in part (b), compute the mean energy  $\langle E \rangle$ .
- **Q4** A quantum system has four single particle states  $|E_0, 1\rangle$ ,  $|E_0, 2\rangle$ ,  $|E_1, 1\rangle$  and  $|E_1, 2\rangle$ , with corresponding single particle energies  $E_0$  and  $E_1$ .
  - (a) List all possible microstates and evaluate the partition function for the system of two identical Fermions which are held at temperature T.
  - (b) Repeat the same for two identical Bosons.

### SECTION B

**Q5** Consider a free classical particle moving in two dimensions  $(q_1, q_2)$  in a box of size L. The classical Hamiltonian of the particle is the following:

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2m} \left( p_1^2 + p_2^2 \right)$$
(2)

where we have  $q_i \in [0, L]$  for i = 1, 2.

(a) Compute in the microcanonical ensemble the accessible area of phase space  $\mathcal{N}(E)$ , defined by

$$\mathcal{N}(E) = \int dp dq \delta(H(q, p) - E) \tag{3}$$

- (b) Compute in the microcanonical ensemble the entropy  $S(E) = k_B \log \Omega(E)$ , where  $\Omega(E) = \frac{N}{N_0}$  and as usual we take  $\mathcal{N}_0 = \frac{h^2}{E}$  with h a constant.
- (c) Compute in the microcanonical ensemble the temperature and the pressure as a function of E. (Note: as the particle moves in two dimensions, the pressure should be viewed as the thermodynamic variable conjugate to the area  $A = L^2$ ). Is your answer consistent with expectations from the ideal gas equation of state? Explain your answer.
- (d) Compute the unconditional probability distribution  $\rho_{unc}(p_1)$  of  $p_1$  in the microcanonical ensemble. You need not normalize it correctly: in this problem we are only asking for the  $p_1$  dependence.
- **Q6** Consider a particle moving in one dimension undergoing a random walk. At each step in the random walk, the particle takes a step where the displacement s is drawn from a normalized probability distribution with density function w(s), where w(s) is a Gaussian of width a:

$$w(s) = \frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{s^2}{2a^2}\right)$$

- (a) Write down the mean  $\langle s \rangle$  and variance  $\langle s^2 \rangle_c$  for a single step.
- (b) Calculate the characteristic function  $\tilde{w}(k)$  associated with the PDF for a single step.
- (c) After N steps, calculate the mean  $\langle X \rangle$  and variance  $\langle X^2 \rangle_c$  of the total displacement from the origin X.
- (d) Calculate the PDF for the total displacement X after N steps, p(X).
- (e) Now consider the modified random walk with the following probability distribution for a single step:

$$w(s) = \frac{1}{\sqrt{2\pi}a} \exp\left(-\frac{(s-\mu)^2}{2a^2}\right)$$

where  $\mu \neq 0$  is a new parameter. For this new choice of w(s), find the mean  $\langle X \rangle$  and variance  $\langle X^2 \rangle_c$  of the total displacement from the origin X after N steps.

**Q7** Consider the system of N non-interacting distinguishable classical particles of mass m moving in one dimension. The particles are subject to a simple harmonic oscillator potential of frequency  $\omega$  as well as a gravitational acceleration g. The resulting potential for the *i*-th particle with linear coordinate  $x_i$  is,

$$V(x_i) = \frac{1}{2} m^2 \omega^2 x_i^2 + mg x_i \,.$$

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The system of the particles is held at fixed temperature T.

- (a) Write down the full Hamiltonian of the system of N particles.
- (b) Compute the partition function of the system  $Z_N$ . **Hint:** You are given that  $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$ .
- (c) Compute the mean energy of the system  $\langle E \rangle$  as well as the heat capacity,

$$C_v = \frac{\partial \langle E \rangle}{\partial T} \,.$$

- (d) Does the system satisfy the equipartition theorem? Should we expect it to satisfy it and why?
- **Q8** Consider a gas of  $\langle N \rangle$  non-interacting Bosons moving inside a *d* dimensional container of volume *V*. The single particle energy is  $E = w k^b$  where *k* is the modulus of the *d*-dimensional wavevector. The constants *b* and *w* are positive numbers.
  - (a) Express the single particle density of states g(E) as a function of the single particle energy E and the volume  $V_d$  of the d-dimensional sphere of unit radius.
  - (b) Write an integral expression for the grand canonical partition function  $\mathcal{Z}$  as a function of the temperature T, the chemical potential  $\mu$ , the constant b and the number of dimensions d.
  - (c) Use the grand canonical partition function  $\mathcal{Z}$  to write an equation relating the density  $\langle N \rangle / V$ , the fugacity z and the temperature T. Your answer should be also expressible in terms of the function,

$$g_y(z) = \frac{1}{\Gamma(y)} \int_0^{+\infty} \frac{x^{y-1}}{z^{-1} e^x - 1} \, dx \, .$$

(d) Given that  $g_y(1)$  is finite only when y > 1, find the necessary condition for Bose-Einstein condensation to happen.