Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH4241-WE01

Title: Representation Theory IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Let (π, V) be a representation of a finite group G. A vector $\boldsymbol{v} \in V$ is said to *generate* the representation if

$$V = \operatorname{span}\{\pi(g)\boldsymbol{v} \,:\, g \in G\}.$$

(a) Show that every irreducible representation (π, V) of G is generated by a non-zero vector $\boldsymbol{v} \in V$.

(b) Give an example of a representation of a finite group that is generated by a non-zero vector, but is not irreducible.

Q2 Let G be the dihedral group of order twelve, $G = \langle r, s | s^2 = r^6 = e,\, sr = r^{-1}s\rangle$, and let H denote the subgroup of G generated by s and r^2: $H = \langle s, r^2\rangle$.

(a) Use the fact that the conjugacy classes of H are $\{e\}$, $\{r^2, r^4\}$, and $\{sr^2, sr^4\}$ to compute $\operatorname{Res}^G_H \chi_\pi$ for each irreducible representation π of G (as listed in the character table below).

size:	1	1	2	2	3	3
	e	r^3	r	r^2	s	sr
$(\mathrm{Id}, \mathbb{C})$	1	1	1	1	1	1
$(\pi_{+-}, \mathbb{C})$	1	1	1	1	-1	-1
$(\pi_{-+}, \mathbb{C})$	1	-1	-1	1	1	-1
$(\pi_{--}, \mathbb{C})$	1	-1	-1	1	-1	1
$(\rho_1, \mathbb{C}^2)$	2	-2	1	-1	0	0
$(\rho_2, \mathbb{C}^2)$	2	2	-1	-1	0	0

The character table of G.

(b) Define a representation $(\rho, \mathbb{C}^2)$ of H by letting

$$\rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(r^2) = \begin{pmatrix} \cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{pmatrix}.$$

Use Frobenius reciprocity to decompose the representation $\operatorname{Ind}^G_H(\rho, \mathbb{C}^2)$ into irreducible representations of G.

Q3 (a) Give a characterization of the Lie algebra of a linear Lie group in terms of the exponential function.

(b) Let $S \in \mathrm{GL}_n(\mathbb{R})$ be an invertible matrix and consider the Lie algebra $\mathfrak{o}(S)$ of the generalized orthogonal group $\mathrm{O}(S) := \{g \in \mathrm{GL}_n(\mathbb{R});\, g\, S\, {}^t g = S\}$. Show that

$$\mathfrak{o}(S) = \{X \in \mathfrak{gl}_n(\mathbb{R});\; XS + S\,{}^t X = 0\}$$

$(S^{-1} \exp(X) S = \exp(S^{-1} X S)$ might be useful).

Q4 Consider the action of $\mathrm{SL}_2(\mathbb{R})$ on the space of smooth functions on column vectors $v \in \mathbb{R}^2$ given by

$$(\pi(g)\varphi)(v) = \varphi\left({}^t g v\right).$$

(a) Show that π defines a group representation.

(b) Compute the associated derived Lie algebra action $D\pi(Y)$ for $Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{R})$.

SECTION B

Q5 Let G be the group of order 16 with the presentation

$$G = \langle a, b | a^2 = b^8 = e,\ aba = b^5 \rangle.$$

Denote by H the subgroup of G generated by b, i.e. $H = \{e, b, b^2, \ldots, b^7\}$. Let $\zeta = e^{2\pi i/8}$, and for each $j = 0, 1, 2, \ldots, 7$, let $(\eta_j, \mathbb{C})$ be the irreducible representation of H defined by $\eta_j(b^n) = \zeta^{nj}$.

Choosing coset representatives e and a for G/H, let W be the vector space $W = a\mathbb{C} \oplus e\mathbb{C}$ on which G acts by $\mathrm{Ind}_H^G \eta_j$.

(a) Compute the matrices of $\mathrm{Ind}_H^G \eta_j(a)$ and $\mathrm{Ind}_H^G \eta_j(b)$ with respect to the basis $\{a1, e1\}$ of W.

(b) Show that

$$\chi_{\mathrm{Ind}_H^G \eta_j}(a^m b^n) = \begin{cases} 0 & \text{if } m \not\equiv 0 \pmod 2 \\ \zeta^{nj}(1 + (-1)^{nj}) & \text{otherwise.} \end{cases}$$

(c) For which values of j is the representation $(\mathrm{Ind}_H^G \eta_j, W)$ irreducible?

Q6 Let $\mathcal{P}_2(n)$ denote the set of all subsets of $\{1, \ldots, n\}$ of size two. Define an S_n-action on $\mathcal{P}_2(n)$ by $\sigma \cdot \{i, j\} = \{\sigma(i), \sigma(j)\}$. Let $\big(\lambda, \mathbb{C}(\mathcal{P}_2(n))\big)$ denote the regular representation of S_n on $\mathbb{C}(\mathcal{P}_2(n))$, i.e.

$$\lambda(\sigma)\left(\sum_{S \in \mathcal{P}_2(n)} z_S S\right) = \sum_{S \in \mathcal{P}_2(n)} z_S\, \sigma \cdot S = \sum_{S \in \mathcal{P}_2(n)} z_{\sigma^{-1} \cdot S}\, S$$

for all $\sigma \in S_n$ and $z_S \in \mathbb{C}$.

(a) Define the linear map $T : \mathbb{C}\big(\mathcal{P}_2(n)\big) \to \mathrm{Sym}^2\mathbb{C}^n$ by

$$T\{i, j\} = e_i e_j,$$

where e_m $(1 \le m \le n)$ is the m-th standard basis vector of $\mathbb{C}^n$. Show that T is an isomorphism of S_n-representations between $\big(\lambda, \mathbb{C}(\mathcal{P}_2(n))\big)$ and $(\mathrm{Sym}^2\pi, U)$; here $(\pi, \mathbb{C}^n)$ denotes the permutation representation of S_n on $\mathbb{C}^n$, and $U \subset \mathrm{Sym}^2\mathbb{C}^n$ is the subspace spanned by $\{e_i e_j : i \ne j\}$.

(b) Let $(\mathrm{Sym}^2\pi, V)$ be a subrepresentation of $(\mathrm{Sym}^2\pi, \mathrm{Sym}^2\mathbb{C}^n)$ such that

$$(\mathrm{Sym}^2\pi, \mathrm{Sym}^2\mathbb{C}^n) = (\mathrm{Sym}^2\pi, U) \oplus (\mathrm{Sym}^2\pi, V).$$

Show that $(\mathrm{Sym}^2\pi, V) \cong (\pi, \mathbb{C}^n)$.

(c) Show that

$$2\chi_\lambda(\sigma) = \big(\#\{1 \le i \le n : \sigma(i) = i\} - 1\big)^2 + \big(\#\{1 \le i \le n : \sigma^2(i) = i\}\big) - 1.$$

Q7 (a) Consider $V = T^3(\mathbb{C}^2) = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the third tensor power of the standard representation of $\mathrm{GL}_2(\mathbb{C})$.

Decompose V into irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ and then of $\mathrm{GL}_2(\mathbb{C})$. Give a weight basis for all of them. (Note that since one representation occurs with higher multiplicity your choice is not canonical).

(b) How often does the trivial representation of $\mathrm{SL}_2(\mathbb{C})$ occur in $T^8(\mathbb{C}^2) = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$, the eighth tensor power of the standard representation of $\mathrm{SL}_2(\mathbb{C})$?

Q8 Let $\mathfrak{h} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} ; x, y, z \in \mathbb{R} \right\}$ be the "Heisenberg" Lie algebra. You may assume that $\mathfrak{h}$ has a basis $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ satisfying

$$[X, Y] = Z, \qquad [X, Z] = [Y, Z] = 0.$$

Let (π, V) be an irreducible finite-dimensional Lie algebra representation of $\mathfrak{h}$.

(a) Use Schur's Lemma to show that there exists a scalar $\lambda \in \mathbb{C}$ such that

$$\pi(Z)v = \lambda v \qquad (*)$$

for all $v \in V$.

(b) Show by induction that for all positive integers k we have

$$\pi(X)\pi(Y)^k = \pi(Y)^k \pi(X) + k\pi(Y)^{k-1}\pi(Z).$$

(c) Assume that $(*)$ holds in (1) with $\lambda = 0$. Show that then $\pi(X)$ and $\pi(Y)$ commute. Conclude that π is one-dimensional.

(d) Now assume $\lambda \neq 0$. Let $v \in V$ be an eigenvector of $\pi(X)$. Use (b) to show that if $v, \pi(Y)v, \ldots, \pi(Y)^n v$ are linearly dependent for some positive n, then $v, \pi(Y)v, \ldots, \pi(Y)^{n-1}v$ are also linearly dependent. Obtain a contradiction.

(Hence there are no finite-dimensional irreducible Lie algebra representations of $\mathfrak{h}$ with $\pi(Z) \neq 0$.)