Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH4281-WE01

Title:
Topics in Combinatorics IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 **1.1** Show that the number of sequences of non-negative numbers $a_1, \ldots, a_{2n}$ with $a_1 = 1$, $a_{2n} = 0$, and $a_i - a_{i+1} = \pm 1$ is equal to the n-th Catalan number C_n.

1.2 Denote by $p_k(n)$ the number of Young diagrams $\lambda \vdash n$ with k rows. Show that

$$p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n+k)$$

Q2 (a) Let P be the root poset of a root system of type A_3. Draw the Hasse diagram of P.

(b) Draw the Hasse diagram of the poset of order ideals of P. Identify join-irreducible elements.

Q3 (a) Let $w = 3\,5\,1\,4\,9\,6\,2\,8\,7 \in S_9$. Apply the Robinson-Shensted-Knuth (RSK) algorithm to compute the insertion and recording tableaux P and Q.

(b) Let (P', Q') be standard Young tableaux of shape $l = (4, 3, 2) \vdash 9$, where

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 8 \\ \hline 3 & 4 & 6 \\ \cline{1-3} 7 & 9 \\ \cline{1-2} \end{array} \qquad Q = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 6 & 7 \\ \hline 3 & 4 & 8 \\ \cline{1-3} 5 & 9 \\ \cline{1-2} \end{array}$$

Find $w' \in S_9$ which is taken to the pair (P, Q) by the RSK algorithm.

Q4 Let (G, S) be a Coxeter system, and let $l(g)$ denote the length of $g \in G$.

(a) Let $g \in G$, $s \in S$. Show that $|l(gs) - l(g)| = 1$.

(b) Let $g \in G$, $s, t \in S$. Suppose that $l(gs) = l(g) + 1$ and $l(tg) = l(g) + 1$. Show that either $l(tgs) = l(g) + 2$ or $tgs = g$.

SECTION B

Q5 Let Δ be a root system. Let $(\cdot, \cdot)$ be the dot product, and let $\langle \alpha \mid \beta \rangle = \dfrac{2(\alpha, \beta)}{(\beta, \beta)}$ for $\alpha, \beta \in \Delta$.

(a) Let $\alpha, \beta \in \Delta$ be non-collinear. Show that if $(\alpha, \beta) < 0$ then $\alpha + \beta \in \Delta$, and if $(\alpha, \beta) > 0$ then $\alpha - \beta \in \Delta$.

(b) Show that there exist integers $p, q \geq 0$, such that the set $I = \{k \in \mathbb{Z} \mid \beta + k\alpha \in \Delta\}$ is an interval $[-q, p] \cap \mathbb{Z}$.

(c) In the setting of part (b), let $R = \{\beta + k\alpha \mid k \in I\}$. Show that $r_\alpha(R) = R$. Show that $q - p = \langle \beta \mid \alpha \rangle$.

Q6 (a) Let P be a finite poset, and let $f : P \to P$ be an order-preserving bijection. Show that f^{-1} is also order-preserving.

(b) Show that for infinite posets the statement of part (a) may not hold.

(c) Let P be a poset such that every chain and every antichain is finite. Show that P is finite.
Hint: consider the set of minimal elements of P.

Q7 (a) Compute the number of lattice paths with steps $(1, \pm 1)$ between points $(0, 1)$ and $(2n, 1)$ that do not go below the x-axis.

(b) A Dyck path is *hill-free* if it has no peaks at height 1 (or, equivalently, does not meet x-axis at two points $2k$ and $2k+2$ for any k). Denote the number of hill-free Dyck paths of length $2n$ by F_n, and let $F(x)$ be the (ordinary) generating function of the sequence F_n. Let $C(x)$ be the generating function of Catalan numbers. Show that the generating function of the hill-free Dyck paths whose leftmost peak is at height 2 is $x^2C(x)F(x)$.

(c) Show that the generating function of the hill-free Dyck paths whose leftmost peak is at height 3 is $x^3C(x)^2F(x)$.
Hint: use the relation $xC(x)^2 - C(x) + 1 = 0$.

(d) Use the relation $F(x) = \dfrac{1}{1 - x^2C(x)^2}$ to show that the number of hill-free Dyck paths of length $2n$ whose leftmost peak is at height 2 or 3 is the $(n-1)$-st Catalan number C_{n-1}.
Hint: Show that $x^2C(x)F(x) + x^3C(x)^2F(x) = x(C(x) - 1)$.

Q8 Let Δ be the root system of type F_4. Let Δ_l and Δ_s be the sets of long and short roots of Δ respectively.

(a) Show that Δ_l and Δ_s are root systems and find their types.

(b) Compute the Coxeter number of Δ_l.

(c) Find the exponents of the Weyl group of Δ_l.