Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH42920-WE01

Title:
Functional Analysis and Applications V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Consider the set
$$c_\ell = \left\{ \{a_n\}_{n\in\mathbb{N}} \in \mathbb{C} \;\middle|\; \lim_{n\to\infty} a_n \text{ exists} \right\}.$$
You may use without proof that c_ℓ is a subspace of ℓ_∞.

1.1 We define $L : (c_\ell, \|\cdot\|_\infty) \to \mathbb{C}$ by
$$L\boldsymbol{a} = \lim_{n\to\infty} a_n.$$
Show that L is a linear functional. Moreover, show that L is bounded and
$$\|L\| = 1.$$

1.2 Show that there exists a bounded linear functional $\mathcal{L} : \ell_\infty \to \mathbb{C}$ such that

- $\|\mathcal{L}\| = 1$.
- For any $\boldsymbol{a} \in c_\ell$ we have that $\mathcal{L}\boldsymbol{a} = \lim_{n\to\infty} a_n$.

Q2 Consider the space ℓ_p of complex sequences defined in class.

2.1 Let $1 \le p < \infty$ be given. Show that the sequence $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{N}} = \left\{\frac{1}{n^\alpha}\right\}_{n\in\mathbb{N}}$ is in ℓ_p if and only if $\alpha p > 1$.

2.2 It is known that for any $1 \le s < r < \infty$ we have that $\ell_s \subseteq \ell_r$, which implies that we can equip ℓ_s with two norms: $\|\cdot\|_s$ and $\|\cdot\|_r$. Using the sequence $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}}$ in ℓ_s defined by
$$(\boldsymbol{a}_n)_j = a_{n,j} = \begin{cases} j^{-1/s} & j \le n \\ 0 & j > n \end{cases}$$
show that $\|\cdot\|_s$ and $\|\cdot\|_r$ are not equivalent on ℓ_s.

Q3 Let $k \in C([0,1])$ with $k(y) > 0$ for all $y \in [0,1]$. Let $\mathcal{X}$ be the Banach space $(C([0,1]), \|\cdot\|_\infty)$. Define the linear operator $T : \mathcal{X} \to \mathcal{X}$ by
$$(Tu)(x) = \int_0^x k(y)u(y)\,\mathrm{d}y, \quad x \in [0,1].$$

3.1 Show that T is bounded and that $\|T\| = \int_0^1 k(y)\,\mathrm{d}y$.

3.2 Show that T^{-1} exists on $\mathcal{D}(T^{-1}) = C^1([0,1]) \subseteq \mathcal{X}$.

Q4 Let T be a closed linear operator in a Hilbert space $\mathcal{H}$. Assume that $\mathrm{i} \in \rho(T)$. Define $U : \mathcal{H} \to \mathcal{H}$ by $U = (T+\mathrm{i})(T-\mathrm{i})^{-1}$.

4.1 Show that $U = I + 2\mathrm{i}(T-\mathrm{i})^{-1}$ and $U^* = (T^*-\mathrm{i})(T^*+\mathrm{i})^{-1}$.

4.2 Show that if T is selfadjoint then U is unitary (i.e. $UU^* = U^*U = I$).

4.3 Assume that T is a selfadjoint operator with compact resolvent and eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$. Prove that there exists an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $\mathcal{H}$ consisting of eigenvectors of U with corresponding eigenvalues $\{\mu_n : n \in \mathbb{N}\}$ such that
$$\forall\, x \in \mathcal{H} : \quad Ux = \sum_{n\in\mathbb{N}} \mu_n \langle x, e_n\rangle e_n, \quad \mu_n = \frac{\lambda_n + \mathrm{i}}{\lambda_n - \mathrm{i}}.$$

Hint: Apply the Spectral Theorem to T.

SECTION B

Q5 Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. We say that a sequence of bounded linear operators $\{T_n\}_{n\in\mathbb{N}}$ from $\mathcal{X}$ to $\mathcal{Y}$ is

- uniformly operator convergent to T if $\|T_n - T\| \underset{n\to\infty}{\longrightarrow} 0$.
- strongly operator convergent to T if $T_n x \underset{n\to\infty}{\longrightarrow} Tx$ for any $x \in \mathcal{X}$.
- weakly operator convergent to T if $T_n x \underset{n\to\infty}{\overset{w}{\longrightarrow}} Tx$.

5.1 Show that if $\{T_n\}_{n\in\mathbb{N}}$ is uniformly operator convergent to T then it is strongly operator convergent to T.

5.2 Show that if $\{T_n\}_{n\in\mathbb{N}}$ is strongly operator convergent to T then it is weakly operator convergent to T.

5.3 Show that if $\{T_n\}_{n\in\mathbb{N}}$ is strongly operator convergent to T then

$$\sup_{n\in\mathbb{N}} \|T_n\| < \infty.$$

Q6 Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $T : \mathcal{D}(T) \subseteq \mathcal{X} \to \mathcal{Y}$ be a closed linear operator. In addition, let $S : \mathcal{D}(S) \subseteq \mathcal{X} \to \mathcal{D}(T)$ be a bounded linear operator such that $\mathcal{D}(S)$ is closed.

6.1 Show that if $\{x_n\}_{n\in\mathbb{N}} \in \mathcal{D}(S)$ is such that $x_n \underset{n\to\infty}{\longrightarrow} x$ then $x \in \mathcal{D}(S)$ and

$$Sx_n \underset{n\to\infty}{\longrightarrow} Sx.$$

6.2 Show that the composition $TS : \mathcal{D}(S) \subseteq \mathcal{X} \to \mathcal{Y}$ is a closed operator and conclude that it must be bounded.

Q7 In the Hilbert space $\mathcal{H} = \ell_2(\mathbb{Z})$ define the linear operator $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{H}$ by

$$(T\boldsymbol{x})_n = \begin{cases} \frac{1}{n}x_{n-1}, & n \geq 1, \\ nx_{n-1}, & n \leq -1, \\ x_{-1}, & n = 0, \end{cases} \quad \mathcal{D}(T) = \left\{ \boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C} : \sum_{n=-\infty}^{-1} |nx_{n-1}|^2 < \infty \right\}.$$

7.1 Find the adjoint operator $T^* : \mathcal{D}(T^*) \subseteq \mathcal{H} \to \mathcal{H}$. You are not required to find $\mathcal{D}(T^*)$.

7.2 Show that $\sigma_p(T) = \mathbb{C}\backslash\{0\}$ and find $\sigma(T)$.

Q8 Consider the sesquilinear form $B : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$ given by

$$B(\varphi, u) = \int_{\mathbb{R}} \varphi'(x+1)\overline{u'(x)}\, \mathrm{d}x.$$

8.1 Find a linear operator $T : \mathcal{D}(T) = H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that $\langle \varphi, Tu \rangle = B(\varphi, u)$ for all $u \in H^2(\mathbb{R})$ and all $\varphi \in H^1(\mathbb{R})$.

8.2 Find a function $u \in H^1(\mathbb{R})$ (other than the constant zero function) with $B(u,u) = 0$.

8.3 Is the sesquilinear form B bounded? Is it coercive? Justify your answers.