

# EXAMINATION PAPER

Examination Session: May/June

2023

Year:

Exam Code:

MATH43020-WE01

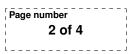
## Title:

# Stochastic Processes V

Time:	3 hours	
Additional Material provided:		
Matariala Davraittadu		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators
Calculators r ennitted.	INO	is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:



### SECTION A

- **Q1** Let  $(\Omega, \mathcal{F}, \mathsf{P})$  be a probability space,  $X : \Omega \to \mathbb{R}$  an integrable random variable, i.e.  $\mathsf{E}[|X|] < \infty$ , and  $\mathcal{G}$  a sub- $\sigma$ -algebra of  $\mathcal{F}$ .
  - (a) State the definition of the abstract conditional expectation  $\mathsf{E}[X|\mathcal{G}]$ .
  - (b) Using (a), show that if  $\mathcal{G} = \{\emptyset, \Omega\}$  is the trivial  $\sigma$ -algebra, then  $\mathsf{E}[X|\mathcal{G}] = \mathsf{E}[X]$  almost surely.
  - (c) Suppose  $\mathsf{E}[X|\mathcal{G}] = \mathsf{E}[X]$  almost surely. Does this imply  $\mathcal{G}$  is the trivial  $\sigma$ -algebra? Give a proof if this is the case, or a counterexample otherwise.
- Q2 This question deals with Poisson processes.
  - (a) Customers arrive at a store according to a Poisson process of rate 10/hour. Each customer is independently a little spender with probability 3/4 or a big spender with probability 1/4. A little spender spends on average 5 pounds and a big spender spends on average 20 pounds. Let T be the total amount of money earned by the shop in the first 5 hours. Find E[T].
  - (b) Consider two independent Poisson processes, one consisting of red balls and the other of blue balls, both having rate  $\lambda$ . Find the probability that 3 red balls appear before 3 blue balls appear.
- Q3 This question deals with martingales.
  - (a) Let  $(\Omega, \mathcal{F}, \mathsf{P})$  be a probability space and let  $X : \Omega \to \mathbb{R}$  be a random variable such that  $\mathsf{E}[|X|] < \infty$ . Let  $\mathcal{F}_n$  for  $n \ge 0$  be a filtration, that is,  $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$  is a nested sequence of  $\sigma$ -algebras. Define  $M_n = \mathsf{E}[X | \mathcal{F}_n]$ . Show that  $M_n$  is a martingale with respect to the filtration  $\mathcal{F}_n$ . Make sure to verify all three martingale conditions.
  - (b) Suppose that  $M_n$  is a martingale with respect to the filtration  $\mathcal{F}_n$ . Suppose also that  $\mathsf{E}[M_n^2] < \infty$  for every n. Show that for i < j,  $\mathsf{E}[(M_j M_i)^2] = \mathsf{E}[M_i^2] \mathsf{E}[M_i^2]$ .

#### SECTION B

Q4 Denote by  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  the set of all non-negative integers. Let U, V be two  $\mathbb{N}_0$ -valued random variables with probability mass functions

$$\mathsf{P}(U=n) = p(1-p)^n$$
 and  $\mathsf{P}(V=n) = e^{-\lambda} \frac{\lambda^n}{n!}$   $\forall n \in \mathbb{N}_0$ 

where  $p \in (0, 1)$  and  $\lambda > 0$ .

- **4.1** Show that if U stochastically dominates V, then  $e^{-\lambda} \ge p$ .
- 4.2 By considering suitable events of a Poisson process or using direct computation, explain why

$$\mathsf{P}(V \ge n) = \mathsf{P}(Z_1 + \dots + Z_n \le \lambda) \qquad \forall n \in \mathbb{N}$$

where  $Z_i \stackrel{i.i.d.}{\sim} \operatorname{Exp}(1)$ . Hence, or otherwise, show that if  $e^{-\lambda} \geq p$ , then U stochastically dominates V.

**4.3** Let W be another random variable with generating function

$$\mathsf{E}[s^W] = \left[\frac{p}{1 - (1 - p)s}\right]^{2023} \qquad \forall s \in (0, 1)$$

with  $e^{-\lambda} \ge p^{2023}$ . Show that W stochastically dominates V. (Hint: what is the generating function for U?)

- **Q5** Consider a renewal process  $M(t) := \sum_{n\geq 0} \mathbb{1}_{\{S_n\leq t\}}$  where  $(S_n)_{n\geq 0}$  is a random walk with delay distribution  $\lambda(t) := \mathsf{P}(S_0 \leq t)$  satisfying  $\mathsf{P}(S_0 \geq 0) = 1$ , and increment distribution  $S_{n+1} S_n \overset{i.i.d.}{\sim}$  Uniform([0,1]). Let  $m(t) := \mathsf{E}[M(t)]$  be the renewal function.
  - **5.1** Derive the renewal equation satisfied by m(t).
  - **5.2** Show that in the zero delay case (i.e.  $P(S_0 = 0) = 1$ ),

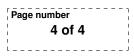
$$m'(t) = \begin{cases} m(t) & \text{for } t \in (0, 1), \\ m(t) - m(t - 1) & \text{for } t > 1, \end{cases}$$

and hence

$$m(t) = \begin{cases} e^t & \text{for } t \in [0, 1], \\ e^t - e^{t-1}(t-1) & \text{for } t \in [1, 2]. \end{cases}$$

**5.3** It is known that for a certain delay distribution  $\lambda$ , the renewal function m(t) is proportional to t for all  $t \ge 0$ . Find a formula for  $\lambda(t)$ .

Please quote any results used and justify all steps carefully in your answer.



**Q6** Let  $X_t$  be a continuous time Markov process on the state space  $\mathcal{I} = \{1, 2, 3\}$  with generator (*Q*-matrix)

$$Q = \begin{pmatrix} -12 & 12 & 0\\ 4 & -10 & 6\\ 0 & 8 & -8 \end{pmatrix}$$

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- 6.1 Show that this Markov process is irreducible.
- **6.2** Find the characteristic polynomial of Q and identify the eigenvalues.
- **6.3** Find the invariant distribution  $\pi$  of the process.
- **6.4** Compute  $p_{2,3}(t)$  exactly.
- **Q7** Let  $X_1, X_2, \ldots$  be independent and identically distributed random variables with common distribution

$$\mathsf{P}(X_i = +1) = \mathsf{P}(X_i = -1) = 1/2.$$

Let  $S_n = \sum_{k=1}^n X_k$  for  $n \ge 1$  and  $S_0 = 0$ . Consider the  $\sigma$ -algebras  $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$  for  $n \ge 1$  and let  $\mathcal{F}_0$  be the trivial  $\sigma$ -algebra.

- **7.1** Let  $M_n = S_n^4 6nS_n^2 + 3n^2 + 2n$  for  $n \ge 0$ . Show that  $M_n$  is a martingale with respect to the filtration  $\mathcal{F}_n$ . Carefully verify all three martingale conditions.
- **7.2** State the definition of a stopping time T with respect to the filtration  $\mathcal{F}_n$ . State any version of the Optional Stopping Theorem. For a positive integer K, define  $T = \inf\{n \ge 0 : |S_n| = K\}$ . Show that T is a stopping time.
- **7.3** For a positive integer K, let  $T = \inf\{n \ge 0 : |S_n| = K\}$ . You may use the fact from lectures that  $\mathsf{E}[T] = K^2$ . Find  $\mathsf{E}[T^2]$ . Carefully justify all steps in your calculation by quoting appropriate theorems.

#### SECTION C

**Q8** Let  $(Z_n^1, Z_n^2)_{n \ge 0}$  be a two-type time-homogeneous branching process with offspring distribution satisfying

$$f^{1}(s_{1}, s_{2}) := \mathsf{E}\left[s_{1}^{Z_{1}^{1}} s_{2}^{Z_{1}^{2}} \middle| (Z_{0}^{1}, Z_{0}^{2}) = (1, 0)\right] = \frac{1}{4}s_{1}s_{2} + ps_{2}^{N} + \left(\frac{3}{4} - p\right)$$
  
and  $f^{2}(s_{1}, s_{2}) := \mathsf{E}\left[s_{1}^{Z_{1}^{1}} s_{2}^{Z_{1}^{2}} \middle| (Z_{0}^{1}, Z_{0}^{2}) = (0, 1)\right] = \frac{1}{8}s_{1}s_{2}^{2} + \frac{1}{4}s_{1}^{2} + \frac{5}{8}$ 

where  $N \in \mathbb{N}$  and  $p \in [0, \frac{3}{4}]$ . Determine the condition on (N, p) under which the process becomes extinct with probability 1.