EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH43320-WE01

Title:
Ergodic Theory V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 (a) When do we call a topological dynamical system (X,T) *uniquely ergodic*? Give an alternative (and equivalent) characterisation of unique ergodicity.

(b) Show that

$$\lim_{n\to\infty} 1/n \cdot \sum_{k=0}^{n-1} \cos(x+k) \;=\; 0$$

for all $x \in \mathbb{R}$.

Q2 Let (X,T) be a topological dynamical system with an invariant measure μ on X. Show that $T\big(\mathrm{supp}(\mu)\big) = \mathrm{supp}(\mu)$.

NB: Recall that

$$\mathit{supp}(\mu) = \{x \in X \colon \textit{for all open sets } U \subseteq X \textit{ with } x \in U, \textit{ we have } \mu(U) > 0\}.$$

You may further use without proof that equivalently, supp(μ) is the unique closed subset of X with $\mu(supp(\mu)) = 1$ and such that if $A \subseteq X$ is closed and $\mu(A) = 1$, then $supp(\mu) \subseteq A$.

Q3 Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \; P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix},$$

and let Σ_A^+ be the associated subshift of finite type.

(a) Is A primitive? Justify your answer.

(b) Evaluate the Markov measure μ_P for P on all cylinders of length 2 for Σ_A^+.

(c) Let α be the state partition for Σ_A^+. Evaluate the information function $I_{\mu_P}(\bigvee_{j=0}^{n-1} \sigma^{-j}\alpha)$.

(d) Evaluate the measure theoretic entropy $h_{\mu_P}(\sigma)$ of μ_P.

Q4 Let (X,d) be a metric space and $T : X \to X$ a continuous map.

(a) State the definition of the topological entropy for (X,T).

(b) State the *Variational Principle* for entropy in the case of a positively expansive map.

(c) (i) Evaluate the topological entropy of $(\mathbb{T}^1, E_3)$.

(ii) Show that the Lebesgue measure is a measure of maximal entropy for $(\mathbb{T}^1, E_3)$. Fully justify your answer.

SECTION B

Q5 Let (X, T) be a topological dynamical system and let $x \in X$. We introduce the set

$$F(x) = \{y \in X \colon \text{there is a strictly increasing sequence } (n_j) \text{ in } \mathbb{N} \text{ with } \lim_{j \to \infty} T^{n_j} x = y\}.$$

NB: In the following, you can take as given that $F(x)$ is always non-empty and compact.

(a) When do we call a subset $A \subseteq X$ *forward invariant*? Show that $F(x)$ is forward invariant.

(b) Show that there is a topological dynamical system (X, T) where there are x and x' in X with $F(x) = X$ and $F(x') = \{x'\}$, respectively.

(c) Is it always true that $TF(x) = F(x)$? Justify your answer.

(d) Define the notion of a minimal dynamical system—you may use any of the several equivalent characterisations from the lectures. Show that (X, T) is minimal if and only if for all $x \in X$, we have $F(x) = X$.

Q6 In the following, we consider $\mathbb{Z}_2 = \{0, 1\}$ equipped with addition mod 2, that is, $0 + 0 = 1 + 1 = 0$ and $1 + 0 = 0 + 1 = 1$. For $\omega_0, \omega_1 \in \mathbb{T}^1$, we define a map

$$T_{\omega_0,\omega_1} \colon \mathbb{Z}_2 \times \mathbb{T}^1 \to \mathbb{Z}_2 \times \mathbb{T}^1, \qquad (i, x) \mapsto (i + 1, x + \omega_i).$$

(a) Given $\omega_0, \omega_1 \in \mathbb{T}^1$, show that there are α_0 and α_1 in $\mathbb{T}^1$ such that

$$\pi \colon \mathbb{Z}_2 \times \mathbb{T}^1 \to \mathbb{Z}_2 \times \mathbb{T}^1, \qquad (i, x) \mapsto (i, x + \alpha_i)$$

is a conjugacy between $(\mathbb{Z}_2 \times \mathbb{T}^1, T)$ and $(\mathbb{Z}_2 \times \mathbb{T}^1, S)$, where $T = T_{\omega_0,\omega_1}$ and $S = T_{(\omega_0+\omega_1)/2,(\omega_0+\omega_1)/2}$. Conclude that $(\mathbb{Z}_2 \times \mathbb{T}^1, T_{\omega_0,\omega_1})$ and $(\mathbb{Z}_2 \times \mathbb{T}^1, T_{\omega_0',\omega_1'})$ are conjugate if $\omega_0 + \omega_1 = \omega_0' + \omega_1'$.

(b) Show that T_{ω_0,ω_1} is minimal if and only if $\omega_0 + \omega_1$ is irrational.

(c) Give a Borel probability measure μ on $\mathbb{Z}_2 \times \mathbb{T}^1$ such that μ is invariant under T_{ω_0,ω_1} for all ω_0 and ω_1 in $\mathbb{T}^1$. Can μ be ergodic under T_{ω_0,ω_1} for all ω_0 and ω_1 in $\mathbb{T}^1$? Justify your answer.
NB: You may use without proof that every Borel set in $\mathbb{Z}_2 \times \mathbb{T}^1$ is of the form $\{0\} \times A \cup \{1\} \times B$ for some Borel sets A, B in $\mathbb{T}^1$.

Q7 Let T be a measure preserving transformation of the probability space $(X, \mathcal{X}, \mu)$.

(a) State the definition of the measure theoretic entropy $h_\mu(T)$.

(b) Suppose that T is invertible. Show that $h_\mu(T) = h_\mu(T^{-1})$.

(c) Let α, β, γ be finite partitions. Show that we have $H_\mu(\alpha \mid \sigma(\gamma)) \leq H_\mu(\alpha \mid \sigma(\beta))$ whenever the partition γ is a refinement of β.
Hint: You may use that

$$H_\mu(\alpha \mid \sigma(\gamma)) = \sum_{A \in \alpha} \int \phi(f_A) d\mu$$

with $\phi(x) = -x \log x$ and $f_A = \mathbb{E}(\mathbb{1}_A \mid \sigma(\gamma))$.

Q8 Let (Σ_A^+, σ) be a subshift of finite type with primitive A.

(a) State the *Gibbs property* for the Parry measure.

(b) Show that the topological entropy of (Σ_A^+, σ) is $\log \lambda$ where λ is the maximal eigenvalue of A.

For $p \in (0,1)$ let $\Lambda(p) \subset [0,1] \backslash \mathbb{Q}$ be the points whose base 2 expansion contains digit 0 with frequency p. That is, $x \in \Lambda(p)$ precisely when $x = \sum_{i=1}^{\infty} a_i 2^{-i}$ with $a_i \in \{0,1\}$ and with $\frac{1}{n}\#\{1 \leq i \leq n : a_i = 0\}$ converging to p. Recall that the t-Hausdorff measure of a set $E \subset \mathbb{R}$ is

$$\mathcal{H}^t(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{i \in \mathbb{N}} |U_i|^t : U_i \text{ open}, E \subset \bigcup_{i \in \mathbb{N}} U_i, |U_i| \leq \delta \right\},$$

where $|U|$ denotes the diameter of a set $U \subset \mathbb{R}$.

(c) Show that $\mathcal{H}^t(\Lambda(p)) = 0$ for any $t > -(p \log p + (1-p)\log(1-p))/\log 2$.