Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH4381-WE01

Title: Topics in Applied Mathematics IV

Time:	3 hours	
Additional Material provided:	Formula sheet.	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Consider the magnetic field $\boldsymbol{B} = \boldsymbol{e}_x + 2x\boldsymbol{e}_y$.

(a) Find a flux function for $\boldsymbol{B}$ and use it to sketch the magnetic field lines.

(b) Compute the magnetic pressure and tension forces for this magnetic field and indicate their directions on your sketch.

(c) Is this $\boldsymbol{B}$ a force-free equilibrium?

Q2 Consider a star with infinite conductivity.

(a) Write down the ideal MHD induction equation and the solenoidal condition.

(b) If the motion inside the star is assumed incompressible ($\nabla \cdot \boldsymbol{u} = 0$), show that

$$(\boldsymbol{B} \cdot \nabla)\boldsymbol{u} = (\boldsymbol{u} \cdot \nabla)\boldsymbol{B}.$$

(c) Now assume that this motion is a rotation of the form $\boldsymbol{u} = r\Omega(r, z)\boldsymbol{e}_\theta$ in cylindrical coordinates, where $\Omega > 0$. If $\boldsymbol{B}(r, z)$ is a steady state, use (b) to show that Ω must be constant along each magnetic field line.

Q3 (a) The Cauchy metric (strain) tensor C has three associated principle invariants: its trace I_1, its sum of principle minors I_2, and its determinant I_3; these can be written in terms of its eigenvalues $(\lambda_1^c, \lambda_2^c, \lambda_3^c)$ as

$$I_1 = \lambda_1^c + \lambda_2^c + \lambda_3^c, \quad I_2 = \lambda_1^c\lambda_2^c + \lambda_1^c\lambda_3^c + \lambda_2^c\lambda_3^c, \quad I_3 = \lambda_1\lambda_2\lambda_3.$$

Briefly describe their geometrical interpretation.

(b) Consider an incompressible deformation (which fixes one of these invariants). State whether it is possible to further fix the value of I_1 as constant and have a non-trivial deformation, **on the assumption the deformation is physically permissible**.

Q4 Consider a deformed tubular body defined as follows:

$$\mathbf{x}(X_1, X_2, X_3) = \mathbf{r}(X_3) + X_1\mathbf{d}_1(X_3) + X_2\mathbf{d}_2(X_3),$$

where $\mathbf{r}$ is some three-dimensional curve. The vector fields $\mathbf{d}_1(X_3)$ and $\mathbf{d}_2(X_3)$ are unit vectors: $\mathbf{d}_i \cdot \mathbf{d}_i = 1$. They span the planes normal to $\mathbf{r}$, that is to say $\mathbf{d}_i \cdot \mathbf{d}_3 = 0$, $i = 1, 2$ where $\mathbf{d}_3 = \mathrm{d}\mathbf{r}/\mathrm{d}X_3$ and $\mathbf{d}_3 \cdot \mathbf{d}_3 = 1$. It is further assumed that $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$.

(a) By assuming that the X_3 derivatives take the form

$$\frac{\mathrm{d}}{\mathrm{d}X_3}\mathbf{d}_i = a_{i1}(X_3)\mathbf{d}_1 + a_{i2}(X_3)\mathbf{d}_2 + a_{i3}(X_3)\mathbf{d}_3,$$

show, by imposing the above stated orthonormality conditions of the basis $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$, that the metric (Cauchy strain) tensor C of the body $\mathbf{x}$ can be written as

$$\mathrm{C} = \begin{pmatrix} 1 & 0 & -u_3X_2 \\ 0 & 1 & u_3X_1 \\ -u_3X_2 & u_3X_1 & (1 - u_2X_1 + u_1X_2)^2 + u_3^2(X_1^2 + X_2^2) \end{pmatrix}$$

and state which of the a_{ij} the functions u_1, u_2, u_3 represent.

(b) Describe the deformation of the body implied by the tensor C.

SECTION B

Q5 Let D be the region $(x, y, z) \in (-1, 1)\times(0, 1)\times\{0\}$, and consider the two-dimensional magnetic field

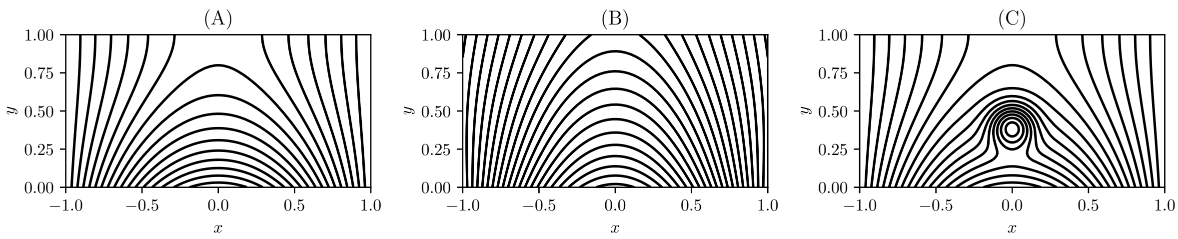
$$\boldsymbol{B}_\mathrm{p} = \cos\left(\frac{\pi x}{2}\right)\sinh\left(\frac{\pi(y-1)}{2}\right)\boldsymbol{e}_x + \sin\left(\frac{\pi x}{2}\right)\cosh\left(\frac{\pi(y-1)}{2}\right)\boldsymbol{e}_y.$$

(a) Show that $\boldsymbol{B}_\mathrm{p}$ is a potential field.

(b) Prove that $\boldsymbol{B}_\mathrm{p}$ is the unique potential field with $B_z = 0$ that satisfies the boundary conditions

$$\text{(i) } B_y(x, 0) = \sin\left(\frac{\pi x}{2}\right)\cosh\left(\frac{\pi}{2}\right), \quad \text{(ii) } B_x(\pm 1, y) = 0, \quad \text{(iii) } B_x(x, 1) = 0.$$

(c) Which of the following three plots shows the magnetic field lines of $\boldsymbol{B}_\mathrm{p}$? Briefly justify your answer.



Q6 Consider a magnetic field of the axisymmetric form

$$\boldsymbol{B} = \nabla \times \left[\frac{A(r,z)}{r}\boldsymbol{e}_\theta\right]$$

in cylindrical coordinates (r, θ, z).

(a) Use Ampère's Law to show that

$$\boldsymbol{J} = \frac{1}{\mu_0}\left[\frac{A}{r^3} - \Delta\left(\frac{A}{r}\right)\right]\boldsymbol{e}_\theta.$$

(b) If $\boldsymbol{B}$ is required to be a (non-zero) magnetostatic equilibrium of the form $\boldsymbol{J} \times \boldsymbol{B} = \nabla p$, show that $p = p(A)$, and further that

$$\frac{J_\theta}{r} = \frac{\mathrm{d}p}{\mathrm{d}A}.$$

(c) Now suppose $p = p_0 + \lambda A$, for $p_0, \lambda \in \mathbb{R}$. By taking the ansatz

$$A(r,z) = f(r)z^2 + g(r),$$

find a magnetostatic equilibrium $\boldsymbol{B}$ that is regular at $r = 0$. (Your solution should involve one free parameter in addition to λ. You need not find p.)

Q7 Consider a thin membrane-like body which can be modelled as a thin elastic sheet model. The central surface $\mathbf{S}$ of the sheet takes the following form:

$$\mathbf{S}(r,\theta) = ar\cos\theta\,\mathbf{E}_1 + ar\sin\theta\,\mathbf{E}_2 + hr^2\sin(n\theta)\,\mathbf{E}_3, \tag{1}$$

where $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ is the Cartesian basis, $r \in [0,1]$ and $\theta \in [0, 2\pi)$, the parameters $a > 0, h > 0$ are real constants, and n is an integer. Its strain energy W is assumed to be given by the Helfrich-type functional

$$W(\mathbf{S}) = \int_{\mathrm{S}} \left(K_m^2 + K_g\right) \mathrm{d}S, \tag{2}$$

where K_m is the mean curvature of $\mathbf{S}$, K_g its Gaussian curvature, and $\mathrm{d}S$ the surface element of $\mathbf{S}$.

(a) Describe the potential range of shapes of this membrane central surface $\mathbf{S}$, paying particular attention to its variation with respect to the constant parameters of the model. You may use sketches.

(b) By considering the shape of the membrane at $r = 0$, explain why an alternative form for the central surface given by

$$\mathbf{S}'(r,\theta) = ar\cos\theta\mathbf{E}_1 + ar\sin\theta\mathbf{E}_2 + hr\sin(n\theta)\mathbf{E}_3$$

would lead to an invalid membrane model, where, by comparison the model (1) is a valid membrane model.

(c) The membrane is found experimentally to be locally compressible, while its overall area is conserved from the state in which $h = 0$. State the functional condition, dependent on the surface metric tensor C_S, which determines this constraint, and state why this constraint would imply h must have a finite value.

(d) It is now assumed $h \ll 1$. To quadratic order in h it can be shown that

$$K_m = -\frac{h\,(n^2-4)\sin(\theta n)}{2a^2}, \quad K_g = \frac{h^2\left[(n^2-4)\cos(2\theta n) - 3n^2 + 4\right]}{2a^4},$$

and that the volume condition in part (c) is satisfied if

$$a^2 = -\frac{4n\left[\pi h^2\left(n^2+4\right) - 8\right]}{32\pi n}$$

(assuming the same order h^2 expansion). Show that the minimum energy of the system has a non-zero n.

Hint: you only need to use the order h^2 part of the energy expression when the constraint is considered.

Q8 Consider a deforming body of the form

$$\mathbf{x}(R,\Theta,Z) = f(\Theta)\left(R\cos\Theta\mathbf{E}_x + R\sin\Theta\mathbf{E}_y\right) + Z\mathbf{E}_z, \tag{3}$$

where $(\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3)$ is the Cartesian basis, $R\in[0,1]$, and $\Theta\in[0,2\pi)$. In its undeformed state $f(\theta)=1$.

(a) Show that the Cartesian form of the Cauchy metric tensor takes the form

$$\mathrm{C} = \begin{pmatrix} f^2 & ff' & 0 \\ ff' & f'^2+f^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $f'(\theta) = \mathrm{d}f/\mathrm{d}\Theta$.

(b) Describe the deformation implied by the tensor C.

(c) The body is modelled as a Hyperelastic material whose Cauchy Stress tensor takes the following form

$$\Sigma = \begin{pmatrix} \frac{-f^2+(f')^2-2}{3f^{10/3}} & \frac{f'}{f^{7/3}} & 0 \\ \frac{f'}{f^{7/3}} & -\frac{f^2+2(f')^2+2}{3f^{10/3}} & 0 \\ 0 & 0 & \frac{-4f^2-2(f')^2+1}{3f^{10/3}} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Give two properties of the model which makes it physically appropriate given that this is a non-linear elastic theory.

(d) Demonstrate that, in the absence of body forces, there is no other equilibrium of the body except one for which the body expands or contracts uniformly ($f'(\theta) = const$).

Hint: In cylindrical coordinates (R,Θ,Z), the components of divergence of a tensor take the form

$$\frac{\partial\Sigma_{RR}}{\partial R} + \frac{1}{R}\frac{\partial\Sigma_{\Theta R}}{\partial\Theta} + \frac{\partial\Sigma_{ZR}}{\partial Z} + \frac{1}{R}\left(\Sigma_{RR}-\Sigma_{\Theta\Theta}\right) \qquad R \text{ componenet}$$

$$\frac{\partial\Sigma_{R\Theta}}{\partial R} + \frac{1}{R}\frac{\partial\Sigma_{\Theta\Theta}}{\partial\Theta} + \frac{\partial\Sigma_{Z\Theta}}{\partial Z} + \frac{1}{R}\left(\Sigma_{R\Theta}+\Sigma_{\Theta R}\right) \qquad \Theta \text{ component}$$

$$\frac{\partial\Sigma_{RZ}}{\partial R} + \frac{1}{R}\frac{\partial\Sigma_{\Theta Z}}{\partial\Theta} + \frac{\partial\Sigma_{ZZ}}{\partial Z} + \frac{\Sigma_{RZ}}{R} \qquad Z \text{ component}$$