EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH1051-WE01

Title: Analysis I

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

Q1 For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ define

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

(a) Show that

$$\binom{\alpha}{k} + \binom{\alpha}{k+1} = \binom{\alpha+1}{k+1}.$$

(b) For $k, n \in \mathbb{N}$ with $n \geq k$ show that

$$\binom{n+1}{k+1} = \sum_{l=k}^{n} \binom{l}{k}.$$

Q2 (a) Determine $\sup(X)$ and $\inf(X)$ for

$$X = \{x \in \mathbb{R} \mid |5x-1| < x+2\},$$

provided they exist. Also, decide whether $\max(X)$ and $\min(X)$ exist.

(b) For $M, N \subset \mathbb{R}$, define

$$M - N = \{m - n \in \mathbb{R} \mid m \in M, n \in N\}.$$

Assume that M, N are bounded and non-empty. Show that $M - N$ is bounded and non-empty, and show that

$$\sup(M-N) = \sup(M) - \inf(N).$$

Justify your statement with results from the lectures.

Q3 Calculate the limits of the following sequences, or show that the limit does not exist. State any results that you use.

(a) $x_n = \dfrac{3^{n+1}-1}{3^n+1}$.

(b) $x_n = \dfrac{1-2+3-4+\cdots-2n}{\sqrt{1+n^2}}$.

(c) $x_n = \left(1+\dfrac{1}{n}\right)^{np}$, where $p \in \mathbb{N}$.

Q4 (a) Determine $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty}$ of the sequence $(x_n)_{n\in\mathbb{N}}$ below. State any results that you use.

$$x_n = \begin{cases} \left(1+\dfrac{1}{n^2}\right)^{n^2} & n = 3k-2, \\ \dfrac{312}{\sqrt[9]{n^8}} & n = 3k-1, \\ (-1)^n \cdot 2 & n = 3k. \end{cases}$$

(b) State the definition of a Cauchy sequence.

(c) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $|x_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Using the definition of a Cauchy sequence, show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

You may use the Theorem of Archimedes.

Q5 Decide whether the following series are convergent. For those that are, decide also whether they are absolutely convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-n}}{n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{(n+1)(n+2)}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n} \cdot n}$.

Q6 (a) Let $x, y \leq 0$. Show that

$$|\exp(x) - \exp(y)| \leq |x - y|.$$

(b) Show that the exponential function is uniformly continuous on $(-\infty, 0]$.

(c) Show that the exponential function is not uniformly continuous on $[0, \infty)$.

Q7 (a) Let $f\colon X \to \mathbb{R}$ be a function with $X \subset \mathbb{R}$ open, and let $c \in \mathbb{R}$. Show that f is differentiable at c if and only if there exists a function $f_1\colon X \to \mathbb{R}$ (depending on c) such that

$$f(x) = f(c) + (x - c)f_1(x)$$

for all $x \in X$ and so that f_1 is continuous at $x = c$.

You may only use results from the lecture notes up to the definition of differentiability.

(b) Show that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt[3]{x}$ is differentiable on $\mathbb{R} \setminus \{0\}$.

(c) Show that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$.

You do not need to use part (a) for parts (b) or (c).

Q8 For each of the following power series, determine the set of all $x \in \mathbb{R}$ for which it converges. State explicitly which results from the lectures you are using.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{e^k + e^{-k}}{2}(x-3)^k.$

(b) $\displaystyle\sum_{k=0}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k+1}\right) x^k.$

Q9 Let $f\colon [0,2] \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x & x \in [0,1] \\ 2-x & x \in (1,2] \end{cases} .$$

Let $f_1\colon \mathbb{R} \to \mathbb{R}$ be the 2-periodic extension of f, that is, if $x \in [2k, 2k+2)$ with $k \in \mathbb{Z}$, then $f_1(x) = f(x-2k)$. Furthermore, for $n \in \mathbb{N}$ and $x \in \mathbb{R}$ let

$$f_{n+1}(x) = \frac{1}{2} f_n(2x).$$

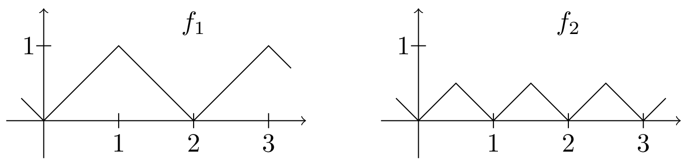

Figure 1: The graphs of f_1 and f_2.

(a) For $k \in \mathbb{N}$ let

$$g_k(x) = \sum_{n=1}^{k} f_n(x).$$

Show that $(g_k)_{k \in \mathbb{N}}$ uniformly converges to a continuous function $g\colon \mathbb{R} \to \mathbb{R}$.

(b) Calculate

$$\int_0^2 g(x)\,dx = \int_0^2 \sum_{n=1}^{\infty} f_n(x)\,dx$$

State all results from the lecture notes that you use.

Q10 Recall that the *floor function* $[\cdot]\colon \mathbb{R} \to \mathbb{Z}$ is defined by $[x] = k$, where $k \in \mathbb{Z}$ is the unique integer with $k \leq x < k+1$.

For $n \in \mathbb{N}$ let $f_n\colon [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \left(\frac{[nx]}{n}\right)^2 .$$

(a) Show that f_n is a step function.

(b) Calculate

$$I_n = \int_0^1 f_n(x)\,dx$$

and determine $\lim_{n\to\infty} I_n$.

Hint: you may want to use that $1 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$.

(c) Show that

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} I_n.$$