Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH1071-WE01

Title:
Linear Algebra I

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

Q1 Let $\Pi \subset \mathbb{R}^3$ be the plane passing through the points

$$\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix}, \text{ and } \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Find an equation defining Π in the form

$$ax + by + cz = d.$$

Q2 Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 2, \\ x_1 + 4x_2 + x_3 - 2x_4 &= 0, \\ 2x_1 + 5x_2 + 5x_3 + 5x_4 &= t. \end{aligned}$$

(a) Find all values of t for which there exist solutions to the linear system.

(b) For these values of t, give the set of solutions to the linear system.

Q3 Find the inverse of the matrix

$$\begin{pmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 3\\0 & 0 & 1 & 2\\0 & 0 & 0 & 1\end{pmatrix}.$$

Q4 Consider the linear map

$$\phi\colon \mathbb{R}[x]_4 \longrightarrow \mathbb{R}^3$$

given by

$$\phi\colon f \longmapsto \begin{pmatrix}f(1)\\f'(0)\\f''(0)\end{pmatrix},$$

where we write $\mathbb{R}[x]_4$ for the vector space of real polynomials of degree at most 4.

(a) Show that ϕ is surjective and hence determine the rank of ϕ.

(b) Find a basis for $\ker(\phi)$.

Q5 Let

$$\boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$$

and consider the map

$$\phi \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

given by

$$\phi \colon \boldsymbol{v} \longmapsto 5\boldsymbol{v} - (\boldsymbol{a} \cdot \boldsymbol{v})\boldsymbol{a}$$

where we write $\cdot$ for the usual dot product.

(a) Show that ϕ is a linear map.

(b) Write down the matrix of ϕ with respect to the standard basis of $\mathbb{R}^3$.

(c) Hence or otherwise give bases for the kernel $\ker(\phi)$ and the image $\mathrm{im}(\phi)$.

Q6 Let A be the matrix

$$A = \begin{pmatrix} 4 & 5 \\ 4 & 3 \end{pmatrix}.$$

Find an invertible matrix M such that $M^{-1}AM = D$ with D diagonal, and use your result to find a matrix B such that $B^3 = A$.

Q7 Let $V = M_n(\mathbb{R})$ be the vector space of all $n \times n$ real matrices, and let T be the linear mapping which sends $A \in V$ to its transpose, $T(A) = A^t$. Define P_+ and P_- by $P_\pm = \frac{1}{2}(1 \pm T)$, where 1 denotes the identity mapping. Show that $P_\pm = P_\pm^2$. What does this mean for the possible eigenvalues of P_+ and P_-?

Q8 Let $V = M_n(\mathbb{C})$ be the vector space of all $n \times n$ complex matrices, and define $\langle\, , \rangle \colon V \times V \to \mathbb{C}$ by

$$\langle A, B \rangle = \mathrm{Tr}(B^* A)$$

where $B^* = \overline{B}^t$ denotes the Hermitian conjugate of B, and Tr denotes the trace, so that $\mathrm{Tr}(A) = \sum_{i=1}^n A_{ii}$.

(a) Show that $\langle\, , \rangle$ defines a complex (Hermitian) inner product on V.

(b) Suppose $C \in V$ is Hermitian and $D \in V$ is anti-Hermitian, so that $C^* = C$ and $D^* = -D$. Show that $\langle C, D \rangle$ is purely imaginary.

Q9 (a) Let $\mathbb{R}[t]_2$ be the real vector space of polynomials $p(t) = p_2t^2 + p_1t + p_0$ of degree at most 2, considered on the interval $[0, 1]$. For p, $q \in \mathbb{R}[t]_2$ set

$$(p, q) = \int_0^1 p(t)q(t)\, dt\,.$$

Find the subspace of $\mathbb{R}[t]_2$ which is orthogonal to the constant function $f(t) = 1$ with respect to this inner product, and state its dimension.

(b) Now we let $C[0, 1]$ be the vector space of *all* continuous real-valued functions $f\colon\ [0, 1] \to \mathbb{R}$. For f, $g \in C[0, 1]$, define

$$(f, g) = \int_0^1 (1 + at)f(t)g(t)\, dt$$

where a is a real constant. For what values of a does this define an inner product on $C[0, 1]$?

Q10 Define the unitary group $U(n)$, and show that it is indeed a group (you can assume the associativity property of matrix multiplication).

Next, define the special unitary group $SU(n)$, and show that it is a subgroup of $U(n)$.

Finally let

$$V(n) = \big\{U \in U(n) \text{ such that } \det(U) \in \{1, i, -i\}\big\}$$

and

$$W(n) = \big\{U \in U(n) \text{ such that } \det(U) \in \{1, -1, i, -i\}\big\}\,.$$

For each of $V(n)$ and $W(n)$, state whether or not they are a subgroup of $U(n)$, and justify your answers.