Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH1081-WE01

Title:
Calculus I (Maths Hons)

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

Q1 **1.1** State the definition of a function $f(x)$ being continuous at a point $x = a$, and use this definition to show that $f(x)$ is continuous at $x = 0$, where $f(x)$ is the function

$$f(x) = \begin{cases} (1-x^2)^2 & x \leq 0 \\ \cos x + x & 0 < x. \end{cases}$$

1.2 Find the global extremal values of $f(x)$ on the interval $[-2, 2]$.

Q2 **2.1** Calculate the Jacobian for the change of variable from Cartesian coordinates $(x, y) \in \mathbb{R}^2$, to the coordinates (u, v), where

$$x = \frac{v-u}{4}, \qquad y = \frac{v+u}{2}.$$

2.2 Let D be the triangle in the xy-plane with vertices $(1/2, 1)$, $(0, 2)$, $(-1/2, 1)$. Using the change of variables from part **2.1**, calculate the double integral

$$\iint_D \frac{y+2x}{y-2x}\, dx\, dy.$$

Q3 **3.1** Find the specific solution $y(x)$ to the initial value problem

$$y'' - 2y' + y = x(x-3), \qquad y(0) = 0, \quad y'(0) = 2.$$

3.2 Find the general solution $y(x)$ to the ordinary differential equation

$$y'' - 2y' + y = 6e^x.$$

Q4 **4.1** Calculate the 3^{rd} order Taylor polynomial for $f(x) = \exp(x^2)$ around $x = 0$.

4.2 Use the Taylor series of $\exp(x^2)$ around $x = 0$ to calculate

$$\lim_{x \to 0} \frac{\exp(2x^2) - \sqrt{1+4x^2}}{x^3}.$$

Q5 **5.1** Let $f(x)$ be periodic with period 2π, and be given for $x \in (-\pi, \pi)$ as

$$f(x) = \begin{cases} \pi & -\pi < x \leq 0, \\ x & 0 < x \leq \pi. \end{cases}$$

Calculate the Fourier series of $f(x)$.

5.2 By considering the Fourier series of $f(x)$ at $x = 0$, compute the value of

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Q6 (a) The function $F(x, y)$ is given as

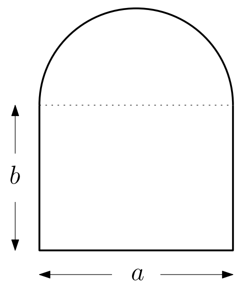
$$F(x, y) = x^2 \ln(x + y) + \cos(y).$$

Use the Chain Rule to calculate the total derivative $\frac{dF}{dx}$ along the curve $y = x^2 - 1$ at $x = 1$.

(b) Find and classify the stationary points of the function

$$f(x, y) = 2xy + y - 2x^2 - 2x - \frac{y^4}{4}.$$

Q7 A two dimensional shape consists of the union of a rectangle of sides a and b and a semicircle of diameter a as shown in the figure below. Find expressions for the area A and perimeter P of this shape in terms of a and b. Use the method of Lagrange multipliers to find the maximum area this shape can have, given that it has perimeter $P = 10$.



Q8 The function $y(x)$ satisfies the differential equation

$$(x^2 - 4)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \lambda y = 0.$$

(a) Explain what is meant by a *singular point* of a linear second order differential equation. Find any singular points in the above differential equation.

(b) The solution to the differential equation can be written in the form of a power series $y = \sum_{n=0}^{\infty} a_n x^n$. Find a recurrence relation between the coefficients a_n.

(c) Find the value of λ for which $a_4 = 0$ but a_2 is not equal to zero in the above power series solution. For this value of λ find a polynomial solution to the differential equation of the form $P_2(x) = 1 + \beta x^2$, where you should determine the value of the constant β.

(d) Show that the integral $\int_{-2}^{2} P_2(x)dx = 0$. Given that the differential operator

$$\mathcal{L} = (x^2 - 4)\frac{d^2}{dx^2} + 2x\frac{d}{dx}$$

is self-adjoint with respect to the inner product $(f(x), g(x)) = \int_{-2}^{2} f(x)g(x)dx$, explain briefly why you would expect this result.

Q9 A bar of metal lies along the x-axis between $x = 0$ and $x = \pi$. The temperature of the bar is denoted $u(x,t)$ and satisfies the heat equation $u_t = k^2 u_{xx}$. The bar is held at a fixed temperature at $x = 0$ so that $u(0,t) = 0$ and is insulated at $x = \pi$ so that $u_x(\pi, t) = 0$.

(a) Show that the series

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\left((n+1/2)x\right) e^{-k^2(n+1/2)^2 t}$$

is a solution to the heat equation, satisfying the conditions $u(0,t) = 0$ and $u_x(\pi,t) = 0$.

(b) Find the coefficients A_n in the case that initially the temperature of the bar is $u(x,0) = x$ for $x \in (0,\pi)$. You may find it helpful to use that

$$\int_0^{\pi} \sin\left((n+1/2)x\right)\sin\left((m+1/2)x\right) = \begin{cases} 0 \text{ for } n \neq m \\ \frac{\pi}{2} \text{ for } n = m. \end{cases}$$

(c) Find an approximation to the time t_c it takes for the temperature at the right hand side of the bar to reach $u(\pi, t_c) = 0.1$.

Q10 (a) Define the Fourier transform $\tilde{f}(p)$ of the function $f(x)$.

(b) A series of functions $f_n(x)$ are defined in terms of $f(x)$ as

$$f_n(x) = \frac{1}{b^n} f(x-n),$$

where b is a constant greater than one. Use the Shift theorem to write the Fourier transform of $f_n(x)$ in terms of $\tilde{f}(p)$, the Fourier transform of $f(x)$.

(c) In the case that

$$f(x) = \begin{cases} 1 \text{ for } 0 < x \leq 1 \\ 0 \text{ otherwise,} \end{cases}$$

find its Fourier transform $\tilde{f}(p)$.

(d) A function $F(x)$ is defined to be

$$F(x) = \begin{cases} b^{-n} \text{ for } n < x \leq n+1 \text{ for each integer } n \text{ greater than or equal to zero} \\ 0 \text{ for } x \leq 0, \end{cases}$$

where b is a constant greater than one. Sketch the function $F(x)$ and find a closed expression (i.e. not as a series) for its Fourier transform $\tilde{F}(p)$.