

EXAMINATION PAPER

Examination Session: May/June Year: 2024

Exam Code:

MATH1091-WE01

Title:

Linear Algebra I (Maths Hons)

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators
		is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:

Q1 Let $\Pi \subset \mathbb{R}^3$ be the plane passing through the points

$$\begin{pmatrix} 1\\0\\2 \end{pmatrix}, \begin{pmatrix} 3\\1\\0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1\\1\\1 \end{pmatrix}.$$

Find an equation defining Π in the form

$$ax + by + cz = d.$$

 ${\bf Q2}\,$ Consider the system of linear equations

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 2,$$

$$x_1 + 4x_2 + x_3 - 2x_4 = 0,$$

$$2x_1 + 5x_2 + 5x_3 + 5x_4 = t.$$

- (a) Find all values of t for which there exist solutions to the linear system.
- (b) For these values of t, give the set of solutions to the linear system.
- ${\bf Q3}\,$ Find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

 ${\bf Q4}$ Consider the linear map

$$\phi \colon \mathbb{R}[x]_4 \longrightarrow \mathbb{R}^3$$

given by

$$\phi \colon f \longmapsto \begin{pmatrix} f(1) \\ f'(0) \\ f''(0) \end{pmatrix},$$

where we write $\mathbb{R}[x]_4$ for the vector space of real polynomials of degree at most 4.

- (a) Show that ϕ is surjective and hence determine the rank of ϕ .
- (b) Find a basis for $\ker(\phi)$.

Q5 Let

$$oldsymbol{a} = egin{pmatrix} 2 \ 0 \ 1 \end{pmatrix} \in \mathbb{R}^3$$

and consider the map

$$\phi \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

given by

$$\phi: \boldsymbol{v} \longmapsto 5\boldsymbol{v} - (\boldsymbol{a} \cdot \boldsymbol{v})\boldsymbol{a}$$

where we write \cdot for the usual dot product.

- (a) Show that ϕ is a linear map.
- (b) Write down the matrix of ϕ with respect to the standard basis of \mathbb{R}^3 .
- (c) Hence or otherwise give bases for the kernel $\ker(\phi)$ and the image $\operatorname{im}(\phi)$.
- **Q6** Let A be the matrix

$$A = \begin{pmatrix} 4 & 5 \\ 4 & 3 \end{pmatrix}.$$

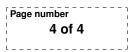
Find an invertible matrix M such that $M^{-1}AM = D$ with D diagonal, and use your result to find a matrix B such that $B^3 = A$.

- **Q7** Let $V = M_n(\mathbb{R})$ be the vector space of all $n \times n$ real matrices, and let T be the linear mapping which sends $A \in V$ to its transpose, $T(A) = A^t$. Define P_+ and P_- by $P_{\pm} = \frac{1}{2}(1 \pm T)$, where 1 denotes the identity mapping. Show that $P_{\pm} = P_{\pm}^2$. What does this mean for the possible eigenvalues of P_+ and P_- ?
- **Q8** Let $V = M_n(\mathbb{C})$ be the vector space of all $n \times n$ complex matrices, and define $\langle , \rangle : V \times V \to \mathbb{C}$ by

$$\langle A, B \rangle = \operatorname{Tr}(B^*A)$$

where $B^* = \overline{B}^t$ denotes the Hermitian conjugate of B, and Tr denotes the trace, so that $\operatorname{Tr}(A) = \sum_{i=1}^n A_{ii}$.

- (a) Show that \langle , \rangle defines a complex (Hermitian) inner product on V.
- (b) Suppose $C \in V$ is Hermitian and $D \in V$ is anti-Hermitian, so that $C^* = C$ and $D^* = -D$. Show that $\langle C, D \rangle$ is purely imaginary.





Q9 (a) Let $\mathbb{R}[t]_2$ be the real vector space of polynomials $p(t) = p_2 t^2 + p_1 t + p_0$ of degree at most 2, considered on the interval [0, 1]. For $p, q \in \mathbb{R}[t]_2$ set

$$(p,q) = \int_0^1 p(t)q(t) dt$$
.

Find the subspace of $\mathbb{R}[t]_2$ which is orthogonal to the constant function f(t) = 1 with respect to this inner product, and state its dimension.

(b) Now we let C[0, 1] be the vector space of *all* continuous real-valued functions $f: [0, 1] \to \mathbb{R}$. For $f, g \in C[0, 1]$, define

$$(f,g)=\int_0^1(1+at)f(t)g(t)\,dt$$

where a is a real constant. For what values of a does this define an inner product on C[0, 1]?

Q10 Define the unitary group U(n), and show that it is indeed a group (you can assume the associativity property of matrix multiplication).

Next, define the special unitary group SU(n), and show that it is a subgroup of U(n). Finally let

$$V(n) = \left\{ U \in U(n) \text{ such that } \det(U) \in \{1, i, -i\} \right\}$$

and

 $W(n) = \left\{ U \in U(n) \text{ such that } \det(U) \in \{1, -1, i, -i\} \right\}.$

For each of V(n) and W(n), state whether or not they are a subgroup of U(n), and justify your answers.