

## **EXAMINATION PAPER**

Examination Session: May/June

2024

Year:

Exam Code:

MATH1551-WE01

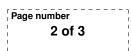
Title:

## Maths For Engineers and Scientists

Time:	3 hours	
Additional Material provided:	Formula sheet.	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:





Q1 Consider the matrices

$$A_1 = \begin{pmatrix} 3 & 2 & 0 \\ 1 & -4 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 2 & 3 \end{pmatrix}.$$

- (a) Determine which of  $A_1$ ,  $A_2$  and  $A_3$  are strictly diagonally dominant and which are positive definite. Justify your answer.
- (b) Consider the system  $A\mathbf{x} = \mathbf{b}$ , where A is taken in turn to be each of  $A_1$ ,  $A_2$  and  $A_3$ .
  - (i) For which of  $A_1$ ,  $A_2$  and  $A_3$  is the Jacobi iterative method guaranteed to converge for any starting values,  $\boldsymbol{x}^{(0)}$ ? Justify your answer.
  - (ii) For which of  $A_1$ ,  $A_2$  and  $A_3$  is the Gauss-Seidel iterative method guaranteed to converge for any starting values,  $\boldsymbol{x}^{(0)}$ ? Justify your answer.
- (c)  $A_3$  can be factorised as  $A_3 = LU$ , where L is a lower triangular matrix with ones on the diagonal and U is an upper triangular matrix. Find L and U.

**Q2** (a) Consider the complex number 
$$z = \frac{2+4i}{2-i} - \frac{4-2i}{1+i}$$
.

- (i) Find the real and imaginary parts of z.
- (ii) Calculate the modulus and principal argument of z. Give the argument in terms of arctan.
- (b) Find all complex numbers, z, satisfying  $\tan(z) = bi$ , where b > 1 is a real number greater than 1. Express your answers in the form z = x + iy where  $x, y \in \mathbb{R}$ .

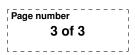
**Q3** (a) (i) Show that the vectors 
$$\boldsymbol{v_1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\1\\1 \end{pmatrix}$$
,  $\boldsymbol{v_2} = \frac{1}{\sqrt{6}} \begin{pmatrix} -2\\1\\1 \end{pmatrix}$ ,  $\boldsymbol{v_3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\1\\-1 \end{pmatrix}$  form an orthonormal basis for  $\mathbb{R}^3$ .  
(ii) Find the coordinates of  $\boldsymbol{u} = \begin{pmatrix} -1\\4\\0 \end{pmatrix}$  with respect to this basis.

(b) Consider the two lines given by

$$\boldsymbol{x} = \begin{pmatrix} 2\\ -3\\ 1 \end{pmatrix} + t \begin{pmatrix} 2\\ 4\\ -2 \end{pmatrix}$$
 and  $(\boldsymbol{x} - \boldsymbol{a}) \times \boldsymbol{d} = \boldsymbol{0},$ 

where  $\boldsymbol{a} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$  and  $\boldsymbol{d} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ .

- (i) Are these two lines parallel? Explain your reasoning.
- (ii) Find the shortest distance between the two lines.



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Q4 (a) Calculate the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}.$$

- (b) With reference to your answer to part (a), explain whether A is diagonalisable or not.
- Q5 (a) Find the following limits, carefully stating any standard results you use:

$$\lim_{x \to 0} \frac{\sin(x)\sin(4x)}{\sin(2x)\sin(3x)}, \qquad \lim_{x \to 0} \frac{e^{2x} - 1}{\ln(1 + 3x)}, \qquad \lim_{x \to \infty} \left(\sqrt{x^2 + x + 1} - x\right).$$

- (b) Let  $f(x) = x^n$  for some fixed positive integer  $n \ge 1$ . Using the definition of the derivative as a limit, show that f(x) is differentiable for all x and find its derivative f'(x).
- Q6 (a) Find Cartesian equations for the tangent plane and normal line to the surface

$$z = \ln(x^2 + 2y^2)$$

at the point (1, 0, 0).

(b) Find all critical points of the function

$$f(x,y) = x^3 - 3x + \sin^2 y$$

and classify each as a local minimum, local maximum or saddle point.

- Q7 (a) Find the degree 2 Taylor polynomial  $P_{2,0}(x)$  of the function  $f(x) = e^{-2x} \cos(2x)$ about x = 0. Estimate the maximum error  $|f(x) - P_{2,0}(x)|$  over the range of values  $0 \le x \le 1/10$ .
  - (b) Show that the differential equation

$$(x^{2}+1)\cos y \,\frac{dy}{dx} + 3x^{2} + 2x\sin y = 0$$

is exact and find the general solution for y(x) as a function of x.

**Q8** The position y(t) of a particle at time  $t \ge 0$  is known to satisfy

$$y'' + cy' + 4y = 0$$

with initial velocity y'(0) = -12 and initial acceleration y''(0) = 0.

- (a) In the case c = 5, solve the equation to find y(t) and calculate the time  $t \ge 0$  when the acceleration y''(t) is maximal.
- (b) Repeat part (a) with c = 4 instead.