Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH1551-WE01

Title:
Maths For Engineers and Scientists

Time:	3 hours	
Additional Material provided:	Formula sheet.	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

Q1 Consider the matrices

$$A_1 = \begin{pmatrix} 3 & 2 & 0 \\ 1 & -4 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 2 & 3 \end{pmatrix}.$$

(a) Determine which of A_1, A_2 and A_3 are strictly diagonally dominant and which are positive definite. Justify your answer.

(b) Consider the system $A\boldsymbol{x} = \boldsymbol{b}$, where A is taken in turn to be each of A_1, A_2 and A_3.

(i) For which of A_1, A_2 and A_3 is the Jacobi iterative method guaranteed to converge for any starting values, $\boldsymbol{x}^{(\boldsymbol{0})}$? Justify your answer.

(ii) For which of A_1, A_2 and A_3 is the Gauss-Seidel iterative method guaranteed to converge for any starting values, $\boldsymbol{x}^{(\boldsymbol{0})}$? Justify your answer.

(c) A_3 can be factorised as $A_3 = LU$, where L is a lower triangular matrix with ones on the diagonal and U is an upper triangular matrix. Find L and U.

Q2 (a) Consider the complex number $z = \dfrac{2+4i}{2-i} - \dfrac{4-2i}{1+i}$.

(i) Find the real and imaginary parts of z.

(ii) Calculate the modulus and principal argument of z. Give the argument in terms of arctan.

(b) Find all complex numbers, z, satisfying $\tan(z) = bi$, where $b > 1$ is a real number greater than 1. Express your answers in the form $z = x + iy$ where $x, y \in \mathbb{R}$.

Q3 (a) (i) Show that the vectors $\boldsymbol{v_1} = \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{v_2} = \dfrac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{v_3} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ form an orthonormal basis for $\mathbb{R}^3$.

(ii) Find the coordinates of $\boldsymbol{u} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$ with respect to this basis.

(b) Consider the two lines given by

$$\boldsymbol{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} \quad \text{and} \quad (\boldsymbol{x} - \boldsymbol{a}) \times \boldsymbol{d} = \boldsymbol{0},$$

where $\boldsymbol{a} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

(i) Are these two lines parallel? Explain your reasoning.

(ii) Find the shortest distance between the two lines.

Q4 (a) Calculate the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}.$$

(b) With reference to your answer to part (a), explain whether A is diagonalisable or not.

Q5 (a) Find the following limits, carefully stating any standard results you use:

$$\lim_{x\to 0} \frac{\sin(x)\sin(4x)}{\sin(2x)\sin(3x)}, \qquad \lim_{x\to 0} \frac{e^{2x}-1}{\ln(1+3x)}, \qquad \lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right).$$

(b) Let $f(x) = x^n$ for some fixed positive integer $n \geq 1$. Using the definition of the derivative as a limit, show that $f(x)$ is differentiable for all x and find its derivative $f'(x)$.

Q6 (a) Find Cartesian equations for the tangent plane and normal line to the surface

$$z = \ln(x^2 + 2y^2)$$

at the point $(1, 0, 0)$.

(b) Find all critical points of the function

$$f(x, y) = x^3 - 3x + \sin^2 y$$

and classify each as a local minimum, local maximum or saddle point.

Q7 (a) Find the degree 2 Taylor polynomial $P_{2,0}(x)$ of the function $f(x) = e^{-2x}\cos(2x)$ about $x = 0$. Estimate the maximum error $|f(x) - P_{2,0}(x)|$ over the range of values $0 \leq x \leq 1/10$.

(b) Show that the differential equation

$$(x^2+1)\cos y \, \frac{dy}{dx} + 3x^2 + 2x\sin y = 0$$

is exact and find the general solution for $y(x)$ as a function of x.

Q8 The position $y(t)$ of a particle at time $t \geq 0$ is known to satisfy

$$y'' + cy' + 4y = 0$$

with initial velocity $y'(0) = -12$ and initial acceleration $y''(0) = 0$.

(a) In the case $c = 5$, solve the equation to find $y(t)$ and calculate the time $t \geq 0$ when the acceleration $y''(t)$ is maximal.

(b) Repeat part (a) with $c = 4$ instead.