Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH1561-WE01

Title:
Single Mathematics A

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

Q1 **1.1** Use the derivative of $\tanh(x)$ to find $\dfrac{d}{dx}(\operatorname{arctanh}(x))$.

1.2 Express the complex number $\left(\dfrac{1+\frac{1}{1+i}}{1+\frac{2}{i}}\right)$ in the form $a+ib$ with a and b real.

1.3 Find $\dfrac{d}{dx}\Big((\cos(x))^{\sinh(x)}\Big)$.

Q2 **2.1** Evaluate the definite integral

$$\int_0^{\pi/2} \cos^3(x)\sin^2(x)\,dx\ .$$

2.2 Find an expression for the definite integral

$$I_n = \int_0^1 e^x (x-1)^n\,dx$$

in terms of I_{n-1}. Use this to find $\displaystyle\int_0^1 e^x(x-1)^4 dx$.

2.3 Evaluate the indefinite integral

$$\int \frac{x^4}{x^2-1}\,dx\ .$$

Q3 **3.1** Evaluate the following limits, you may use any method. You will only get full marks if you explain all the steps and rules you are using to prove the limits:

(a) $\displaystyle\lim_{x\to\infty} \sin(x)e^{-x}$

(b) $\displaystyle\lim_{x\to 1} \frac{(x-1)^2}{\ln(x)\cos(\pi x/2)}$

(c) $\displaystyle\lim_{x\to 0} \sin^2(x)\sin(\tfrac{1}{x})$.

3.2 Find all complex solutions of the equation

$$(e^z+2)e^z = -2$$

in the form $z = x+iy$.

Q4 (a) Determine whether or not the series

$$\sum_{n=1}^{\infty} 4^n e^{-(2n+3)}, \qquad \sum_{n=1}^{\infty}\left(\frac{n+3}{2n+5e^{-n}}\right)^{2n}$$

converge.

(b) Determine the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{1}{n^2\,\ln(n+3)}\,(3x)^n\ .$$

Determine whether the series converges at the endpoints of the interval of convergence.

Q5 **5.1** Let $f(x) = \cos(\ln(x+1))$.

(a) Find the second-order Taylor polynomial $p_2(x)$ of $f(x)$ about $x = 0$.

(b) Use the Lagrange form of the remainder to obtain bounds on the errors

$$\left|f\left(-\frac{1}{2}\right) - p_2\left(-\frac{1}{2}\right)\right| \qquad \text{and} \qquad \left|f\left(\frac{1}{3}\right) - p_2\left(\frac{1}{3}\right)\right|.$$

5.2 (a) Consider the matrix

$$A = \begin{pmatrix} \frac{1}{3} & a & 0 \\ b & -\frac{1}{3} & 0 \\ 0 & 0 & c \end{pmatrix}$$

Find all possible combinations of a, b, c such that the matrix A is orthogonal.

(b) Let B be an orthogonal matrix. Show that B^{-1} is also an orthogonal matrix.

(c) Assume a matrix M is both orthogonal and symmetric. Show that the only possible eigenvalues of M are ± 1. You may use any facts given in the Lectures.

Q6 Consider the following inhomogeneous system of linear equations.

$$\begin{aligned} x + y + kz &= 2 \\ 3x + 4y + 2z &= k \\ 2x + 3y - z &= 1 \end{aligned}$$

(a) For which values of $k \in \mathbb{R}$ does the system of linear equations have (i) no solutions, (ii) a unique solution, (iii) infinitely many solutions?

Find the solutions in cases (ii) and (iii) and, in case (iii), also say whether the solution represents a line or a plane.

(b) Find the values of $k \in \mathbb{R}$ for which the corresponding homogeneous system of linear equations has infinitely many solutions. Find the solutions for these values of k.

Q7 **7.1** Consider the matrix

$$A := \begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 1 \\ 4 & 0 & -2 \end{pmatrix}.$$

Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

7.2 Let

$$B := \begin{pmatrix} 2 & 0 & 0 \\ -20 & 2 & 5 \\ 20 & 0 & -3 \end{pmatrix}.$$

Verify that

$$A \cdot B = B \cdot A.$$

Compute the matrix product $P^{-1}BP$ with the matrix P from Question 7.1.

7.3 Show the following general fact: If $C, D \in \mathrm{Mat}_{n,n}(\mathbb{R})$ commute, that is, $C \cdot D = D \cdot C$ and $v \in \mathbb{R}^n$ is a vector in an eigenspace $V_\lambda(C)$ of the matrix C, then we have

$$Dv \in V_\lambda(C).$$

END