Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH1571-WE01

Title: Single Mathematics B

Time:	3 hours	
Additional Material provided:	Tables: Normal distribution	
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

Q1 (a) Let A, B and C be points with Cartesian coordinates $(1,5,2)$, $(2,7,3)$ and $(2,5,-1)$ respectively. State the parametric equation of the line L passing through points B and C and further compute the distance of the point A from the line L.

(b) A particle of constant mass m moving in a plane follows a trajectory

$$\theta = t\ , \quad r = \frac{10}{\cos\theta(t)}\ ,$$

in polar coordinates as a function of time t.

(i) Compute the trajectory of the particle in Cartesian coordinates, and sketch the path of the particle.

(ii) Compute the velocity and acceleration of the particle.

(iii) Is there a known physical force that could produce this motion? Justify your answer.

Q2 (a) Obtain the solution $x(t)$ of the damped oscillator equation

$$\ddot{x} + 4\dot{x} + 4x = \cos(2t),$$

which satisfies $x(0) = 0$ and $\dot{x}(0) = 4$.

(b) Find the general solution $y(x)$ satisfying the equation

$$\frac{dy}{dx} + 3\tan(x)y = 3\sec(x)y^{2/3}.$$

Q3 Let $g(x)$ be a 2π-periodic function defined by

$$g(x) = x^3 - \pi^2 x, \text{ for } -\pi < x \leq \pi.$$

(a) Compute the Fourier series for g.

(b) Using Parseval's theorem or otherwise, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Q4 Consider the transformation from Cartesian to polar coordinates, given as usual by $x = r\cos(\theta)$, $y = r\sin(\theta)$.

(a) Suppose we have the function $f(x, y) = e^{x^2-y^2}$. If $F(r, \theta)$ is the function representing f after the transformation to polar coordinates, $F(r, \theta) = f(x, y)$, use the multi-variable chain rule on $f(x, y)$ to calculate $\frac{\partial F}{\partial \theta}$, giving your answer in terms of r and θ only.

(b) By first writing r and θ in terms of x and y, show that

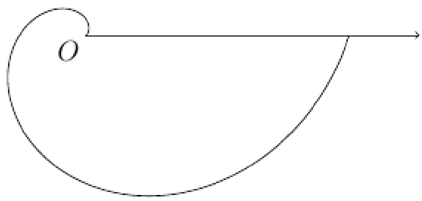
$$\frac{\partial r}{\partial x} = \cos(\theta), \qquad \frac{\partial r}{\partial y} = \sin(\theta),$$
$$\frac{\partial \theta}{\partial x} = -\frac{\sin(\theta)}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{\cos(\theta)}{r}.$$

Hint: You may assume that $\frac{\mathrm{d}}{\mathrm{d}u}(\tan^{-1}(u)) = \frac{1}{1+u^2}$.

(c) The figure below shows a spiral curve with polar equation

$$r = a\theta, \qquad 0 \leq \theta \leq 2\pi,$$

where a is a positive constant. R is the finite region enclosed by the curve and the $\theta = 0$ axis.



Find

$$\iint_R (1 + x^2 + y^2)dA.$$

Q5 (a) Find the critical point(s) of the function

$$f(x,y) = x^3 - 12xy + 48x + by^2$$

when:

(i) the value of the constant b is $b = 2.4$;

(ii) the value of the constant b is $b = -3$.

For each value of b, either classify the critical point(s) or, if the usual second-derivative criterion does not work, explain why not. (You do not have to classify the critical point(s) in this case.)

(b) Consider the function

$$g(x,y) = e^{-(x+2y)^2}.$$

Find the set of its critical points, and evaluate the function at those points. (**Hint:** You should find a one-parameter family of critical points.) Is the usual second-derivative criterion sufficient to classify the critical points? Explain why (or why not), and describe the feature of the surface represented by this set of points.

Q6 (a) Consider the three-dimensional vector field

$$\boldsymbol{V}(x,y,z) = \frac{1}{2}(y-1)\boldsymbol{i} + \frac{1}{2}(1-x)\boldsymbol{j} + z^2\boldsymbol{k}.$$

(i) Sketch the vector field in the plane $z = 0$.

(ii) Calculate the divergence and the curl of $\boldsymbol{V}$. Discuss which components of the vector field contribute to the divergence and curl. How does this relate to your sketch?

(b) Show that:

(i) for any smooth function f we have

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0.$$

(ii) for any vector field

$$\boldsymbol{V} = V_1(x,y,z)\boldsymbol{i} + V_2(x,y,z)\boldsymbol{j} + V_3(x,y,z)\boldsymbol{k},$$

we have

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{V}) = 0.$$

Q7 In 2024, the Olympic Games will be held in Paris. Athletes will travel from all over the world to take part in this competition.

(a) Twenty athletes are travelling from Charles de Gaulle Airport to the Olympic stadium. Three minibuses have arrived to collect them. There are five seats on the first bus, seven seats on the second, and eight on the third.

(i) How many different ways are there for the twenty athletes to divide themselves between the three buses?

(ii) Eight of the athletes are German, and the rest are Spanish. If each athlete is assigned to a bus at random, what is the probability that all eight German athletes are on the largest bus?

(b) Spectators travelling to the Olympics often want to participate in local culture. We will assume that the quantity of cheese, in kilograms, consumed by each spectator has the following probability density function.

$$f_X(x) = \begin{cases} \alpha x^{-3} & \text{if } x \geq 2 \\ 0 & \text{otherwise} \end{cases}.$$

(i) For which value of α is this a valid probability distribution? Sketch the density function.

(ii) What is the mean quantity of cheese eaten by a single spectator?

(iii) Doctors recommend that spectators do not eat more than 6kg of cheese in a short space of time. Calculate the cumulative distribution function $F_X(x)$ and use it to find the probability that a spectator eats too much cheese.