Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH1597-WE01

Title:
Probability I

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

Q1 A committee of 4 people is to be formed out of 10 men and 5 women. Suppose all 4 people are chosen at random.

(a) What is the probability that the committee consists of 2 men and 2 women?

(b) There is a couple, Bob and Anne out of these 15 people. What is the probability that both Bob and Anne are on the committee?

(c) What is the probability that Bob is selected on the committee, but Anne is not selected on the committee?

(d) Now, suppose that it is decided that Jane (who is one of the 5 women) will be on the committee, and the rest of the 3 members are chosen at random. What is the probability that exactly one of Bob or Anne is on the committee?

[Full credits to **Q1** will be given if your answers are in terms of binomial coefficients]

Q2 There are two coins, one of them is fair, and the other one is biased with the probability of head being $3/4$ for the biased coin. The coins are independent of each other and it is equally likely to choose either of the coins. Pick one of the coins and toss it 3 times, and let X denote the number of heads obtained in these 3 tosses.

(a) Write the probability mass function of X, when it is known that the fair coin has been tossed. What is the name that we gave in class to the conditional distribution of X in this case?

(b) Write the probability mass function of X, when it is known that the biased coin has been tossed. What is the name that we gave in class to the conditional distribution of X in this case?

(c) Find $\mathbb{E}[X]$.

(d) Find the probability that the coin chosen is fair, given the information that all three tosses were heads.

Q3 Let X and Y be random variables with joint probability mass function given by the following table :

$p(x,y)$	$y=-1$	$y=0$	$y=2$
$x=-1$	$1/7$	$1/7$	0
$x=0$	$1/7$	$5/14$	$1/14$
$x=2$	c	$1/14$	$1/14$

Furthermore, it is known that $\mathbb{P}(X=x, Y=y) = 0$ for all other values (x,y).

(a) Find c.

(b) Find the marginal probability mass functions $p_X(\cdot)$ and $p_Y(\cdot)$ for X and Y, respectively. Find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$. Find $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.

(c) Find $\mathrm{Cov}(X,Y)$. Are X and Y independent? Justify your answer.

(d) Find $\mathrm{Cov}(X, X+Y)$ and $\mathrm{Var}(X+Y)$.

Q4 Let X be a random variable with probability density function given

$$f(x) = \begin{cases} c|x|, & -1 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find c.

(b) Define $Y := X^2$. Let $M_Y(t)$ denote the moment generating function of Y. Show that $M_Y(t) = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}$ for any $t \neq 0$. [Hint: Use the Taylor expansion $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$]

(c) Prove that

$$\lim_{t \to 0} M_Y(t) = M_Y(0).$$

(d) Find $\mathbb{E}[X^4]$ using $M_Y(t)$. State clearly any property of the moment generating function you use. You will get only partial credit if you do not use $M_Y(t)$ to evaluate $\mathbb{E}[X^4]$.

Q5 Suppose we throw a fair six-sided die n times and let $X_i(1 \leq i \leq n)$, record the score of the i-th throw. We assume that the throws are independent of each other.

(a) Show that $\mu = \mathbb{E}[X_i] = 7/2$ for $1 \leq i \leq n$.

(b) Show that $\sigma^2 = \mathrm{Var}(X_i) = 35/12$, for $1 \leq i \leq n$.

(c) Let $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Show that $\mathbb{E}\left[\overline{X}\right] = 7/2$ and $\mathrm{Var}\left(\overline{X}\right) = 35/12n$. Justify each step.

(d) Assume that $n = 25$. Use the Central Limit Theorem, to show that

$$\mathbb{P}\left(|\overline{X} - 3.5| \geq 0.5\right) \approx 0.144.$$

You can use the fact that $\frac{0.5}{\sqrt{7/60}} = 1.46$. You can use the fact that $\mathbb{P}(Z \leq -1.46) \approx 0.072$, if Z is a standard Normal random variable.