Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH2011-WE01

Title:
Complex Analysis II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

SECTION A

Q1 Let $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be defined by

$$d(z,w) = \begin{cases} 0, & z = w, \\ 2, & z \neq w. \end{cases}$$

1.1 Show that d defines a metric on $\mathbb{C}$.

1.2 Let $z_n = 1/n$ for $n \in \mathbb{N}$. Do we have $\lim_{n\to\infty} z_n = 0$ in the metric space $(\mathbb{C}, d)$? Justify your answer.

Q2 For $n \in \mathbb{N}$ define the function $f_n : \mathbb{C} \to \mathbb{C}$ by

$$f_n(z) = \begin{cases} 2^{-n}, & \mathrm{Re}(z) \geq 0, \\ 3^{-n}, & \mathrm{Re}(z) < 0. \end{cases}$$

2.1 Show that $\sum_{n=1}^{\infty} f_n$ converges uniformly on $\mathbb{C}$ to a function $F : \mathbb{C} \to \mathbb{C}$.

2.2 Is the function F continuous on $\mathbb{C}$? Justify your answer.

Q3 Consider the function $f(z) = \dfrac{2z-9}{(2z+1)(z-2)}$ and recall that the Taylor series given by $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n$ converges for $|z| < 1$.

3.1 Find real constants A, B such that $f(z) = \dfrac{A}{2z+1} + \dfrac{B}{z-2}$.

3.2 Use the Taylor series given above to determine the Taylor series of $\dfrac{1}{z-2}$ around the point $z = 0$. What is its radius of convergence?

3.3 Calculate the Laurent series of $\dfrac{1}{2z+1}$ on the annulus $|z| > 1/2$.

3.4 Hence, find the Laurent series of f on the annulus $1/2 < |z| < 2$.

Q4 Using the 'substitution' $z = e^{i\theta}$, or otherwise, find a meromorphic function f such that

$$\int_0^{2\pi} \frac{\cos(\theta)}{25 - 24\cos(\theta)}\, d\theta \;=\; \int_{|z|=1} f(z)\, dz.$$

Hence, evaluate the integral.

SECTION B

Q5 Define $f : \mathbb{C}\backslash\{i\} \to \mathbb{C}$ by $f(z) = \frac{1}{z-i}$.

5.1 Verify that f satisfies the Cauchy–Riemann equations for all $z \in \mathbb{C}\backslash\{i\}$.

5.2 Find the image $f(D)$ of $D = \{z \in \mathbb{C} : \, |z| < 1, \, \mathrm{Re}(z) > 0\}$.

Q6 Define $f : \mathbb{C}\backslash\{-i\} \to \mathbb{C}$ by $f(z) = \frac{z-i}{z+i}$. For $z_1, z_2 \in \mathbb{C}$, let $L(z_1, z_2)$ denote the straight line segment in $\mathbb{C}$ connecting the point z_1 with the point z_2.

6.1 Let $D = \{z \in \mathbb{C} : \, \mathrm{Im}(z) > -1\}$. Find a holomorphic function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$.

6.2 Consider the line segment $\gamma_1 = L(1, -1)$. Calculate $\int_{\gamma_1} f(z)\, dz$.

6.3 Consider the following contour consisting of three line segments:

$$\gamma_2 = L(-1, -1-2i) \cup L(-1-2i, 1-2i) \cup L(1-2i, 1).$$

Calculate $\int_{\gamma_2} f(z)\, dz$.

Q7 Let $f : D \to \mathbb{C}$ be a non-constant holomorphic function on a non-empty domain $D \subset \mathbb{C}$ that is symmetric about the real axis (that is, $z \in D$ if and only if $\overline{z} \in D$).

7.1 Suppose the Taylor series of f around a point $z_0 \in D$ is given by

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad (c_n \in \mathbb{C}).$$

Consider the function $g(z) := \overline{f(\overline{z})}$. Calculate the Taylor series of g around the point $\overline{z_0}$ in terms of the coefficients c_n. Hence, or otherwise, demonstrate that g is holomorphic on D.

7.2 Explain why the set $D \cap \mathbb{R}$ must be non-empty. Hence, show that $D \cap \mathbb{R}$ contains a non-isolated point.

7.3 Prove that if f takes real values on the set $D \cap \mathbb{R}$ then on D it must satisfy

$$\overline{f(z)} = f(\overline{z}) \qquad (z \in D).$$

State any results from lectures that you use.

7.4 For the case $D = \mathbb{C}$, deduce that if f takes real values on $\mathbb{R}$ then it cannot be bounded on the closed lower half-plane $\{z = x + iy \in \mathbb{C} \, : \, y \leq 0\}$.

Q8 Let g be a meromorphic function on $\mathbb{C}$ with finitely many poles $a_1, a_2, \ldots a_n$ and fix a real number ρ such that $\rho > \mathrm{Re}(a_j)$ for every $1 \leq j \leq n$.

For any real $R > 0$ denote by I_R the straight line segment in $\mathbb{C}$ connecting the point $\rho - iR$ with the point $\rho + iR$ and consider the function $f : \mathbb{R}_{>0} \to \mathbb{C}$ given by

$$f(t) := \frac{1}{2\pi i} \lim_{R\to\infty} \int_{I_R} g(z)\, e^{zt}\, dz \qquad (t > 0).$$

You may assume that this function is independent of the choice of ρ.

8.1 Suppose that there exist real constants $M > 0$, $r > 0$ and $k > 1$ such that

$$|g(z)| \leq M|z|^{-k} \qquad \text{whenever} \qquad |z| \geq r.$$

Use the Estimation Lemma to show that for every $t > 0$ we have

$$\int_{C_R} g(z)\, e^{zt}\, dz \longrightarrow 0 \qquad \text{as} \qquad R \longrightarrow \infty,$$

where C_R is the semi-circular contour $C_R(\theta) = \rho + Re^{i\theta}$, for $\theta \in [\pi/2, 3\pi/2]$, connecting $\rho + iR$ with $\rho - iR$.

8.2 Hence, calculate $f(t)$ in the case that $g(z) = \dfrac{z+1}{z^3 - z^2}$.