Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH2031-WE01

Title:
Analysis in Many Variables II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

SECTION A

Q1 Consider the vector field $\boldsymbol{f}(\boldsymbol{x}) = x^2\,\boldsymbol{e}_1 + \cos y \sin z\,\boldsymbol{e}_2 + \sin y \cos z\,\boldsymbol{e}_3$.

(a) By finding a suitable scalar potential, show that $\boldsymbol{f}(\boldsymbol{x})$ is conservative.

(b) Evaluate $\displaystyle\int_C \boldsymbol{f}\cdot d\boldsymbol{x}$ along the curve C with parametrisation

$$\boldsymbol{x}(t) = (t^2+1)\,\boldsymbol{e}_1 + e^t\,\boldsymbol{e}_2 + e^{2t}\,\boldsymbol{e}_3, \quad \text{for } t \in [0,1].$$

Q2 Compute the surface area of the surface with parametrisation

$$\boldsymbol{x}(u,v) = u\cos v\,\boldsymbol{e}_1 + u\sin v\,\boldsymbol{e}_2 + u^2\,\boldsymbol{e}_3 \quad \text{for } u \in [0,\sqrt{2}],\ v \in [0,2\pi).$$

Q3 (a) Let $a \in \mathbb{R}\text{-}\{0\}$. By integrating against an arbitrary test function show that $(x-a)^2\delta'(x-a) = 0$.

(b) Solve the following equation for the generalised function g,

$$(4x^3 - 8x^2 - 3x + 9)\,g(x) = 0,$$

i.e. find the generalised solution $g(x)$ in terms of shifted delta distributions δ_a and possibly their derivatives. Justify the steps taken to arrive at the solution.

Q4 You are given the linear operator

$$L = x^2 e^x \frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x),$$

with the two real-valued functions $g \in \mathcal{C}^1([1,2])$ and $h \in \mathcal{C}^0([1,2])$.

(a) Calculate the formal adjoint L^* of L as a function of g and h.

(b) Choose g so that L is formally self-adjoint.

(c) Denoting the formally self-adjoint operator found in part (b) as $\mathfrak{L}$, consider the Boundary Value Problem on $[1,2]$ given by

$$\mathfrak{L}u = 0, \qquad u'(1) - u(2) = 0,\ u'(2) = 0.$$

Is this BVP self-adjoint? Justify your answer fully.

SECTION B

Q5 Consider the vector field

$$\boldsymbol{f}(\boldsymbol{x}) = (x^3 + xz + yz^2)\,\boldsymbol{e}_1 + (xyz^3 + y^7)\,\boldsymbol{e}_2 + x^2 z^5\,\boldsymbol{e}_3,$$

and let S be the union of two smooth surfaces S_1 and S_2, where S_1 is defined by

$$x^2 + y^2 = 9, \quad z \in [0, 8],$$

and S_2 is defined by

$$x^2 + y^2 + (z-8)^2 = 9, \quad z \geq 8.$$

(a) Compute $\nabla \times \boldsymbol{f}$.

(b) Sketch the surface S, and state whether it is open or closed.

(c) By using Stokes' Theorem, or otherwise, compute the flux of $\nabla \times \boldsymbol{f}$ through S, taking the surface normal to be *away from the origin*.

Q6 A closed surface, S, is given by the parametrisation

$$\boldsymbol{x}(u, v) = \cos u\,\boldsymbol{e}_1 + \cos v\,\boldsymbol{e}_2 + \cos(u+v)\,\boldsymbol{e}_3, \quad \text{for } u \in [0, \pi],\ v \in [-\pi, \pi].$$

(a) Find a normal vector to S. [This need *not* be a unit vector.]

(b) Using the Divergence Theorem with the vector field $\boldsymbol{f}(\boldsymbol{x}) = x\,\boldsymbol{e}_1$, calculate the volume enclosed by S.

Q7 Let $L = d^2/dx^2$ be the one-dimensional Laplacian.

(a) Consider Poisson's equation $Lu(x) = f(x)$ for complex-valued functions $u \in \mathcal{C}^2([0, \ell])$ and periodic boundary conditions $u(0) = u(\ell)$. The eigenfunctions for this problem may be expressed in the form $u_n(x) := \exp(ianx), n \in \mathbb{Z}$ for some real number a.

(i) Give the value of $a \in \mathbb{R}$ in terms of ℓ and give an expression for the normalised eigenfunctions $\widehat{u_n}(x)$.

(ii) Will Poisson's equation have unique solutions? Why or why not?

(iii) Assume that the source term f admits an eigenfunction expansion. Under what conditions (if any) on f would a solution exist?

(b) If we instead consider the differential equation $Lv(x) = g(x)$ for functions v on $[0, \ell]$ satisfying the boundary condition $v(0) = v(\ell) + 1$, do these functions v form a vector space? Why or why not?

(c) In order to transform the problem of part (b) into the original $Lu(x) = f(x)$ problem with $u(0) = u(\ell)$ of part (a), one could write $u(x) = v(x) + h(x)$ and $f(x) = g(x) + k(x)$ for some functions h and k. Find the conditions that h and k must satisfy to achieve this goal. Give a simple example of a function h.

Q8 (a) Consider the two-dimensional domain

$$D = \mathring{D} \cup \partial D := \{ (r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le \pi/2 \},$$

where $\mathring{D} = \{(r,\theta) : 0 < r < 1,\ 0 < \theta < \pi/2\}$ is the interior of the domain and ∂D its boundary. Denote the origin of the plane by O and label P the point in $\mathring{D}$ with $\boldsymbol{OP} := \boldsymbol{x}_0 = \frac{1}{2}\,\boldsymbol{e}_1 + \frac{1}{2}\,\boldsymbol{e}_2$. Use the method of images to construct the Green's function $G(\mathbf{x}, \mathbf{x_0})$ satisfying

$$\begin{aligned} &\boldsymbol{\nabla}^2\, G(\boldsymbol{x}, \boldsymbol{x_0}) = \delta(\boldsymbol{x} - \boldsymbol{x_0}) && \text{for } \boldsymbol{x} \in D, \\ &G(\boldsymbol{x}, \boldsymbol{x_0}) = 0 && \text{for } \boldsymbol{x} \in \partial D. \end{aligned}$$

You may use the fact that the fundamental solution of Laplace's equation, which is regular on $\mathbb{R}^2 - \{\boldsymbol{x_0}\}$, is given by

$$G_0(\boldsymbol{x}, \boldsymbol{x_0}) = \frac{1}{2\pi} \ln|\boldsymbol{x} - \boldsymbol{x_0}|.$$

Draw a rough sketch indicating the position of the point P and of its images to support your result for the Green's function $G(\boldsymbol{x}, \boldsymbol{x_0})$. Clearly mark the domain D, label your image points as P_i (with $\boldsymbol{OP_i} := \boldsymbol{x_i}$) and call Q the point such that $\boldsymbol{OQ} := \boldsymbol{x}$. Give your answer for the Green's function $G(\boldsymbol{x}, \boldsymbol{x_0})$ in terms of $\boldsymbol{x_0}, \boldsymbol{x}$ and $\boldsymbol{x_i}$.

(b) Prove that the solution $G(\boldsymbol{x}, \boldsymbol{x_0})$ you obtained in part (a) satisfies $G(\boldsymbol{x}, \boldsymbol{x_0}) = 0$ for all points Q with polar coordinates $(r,\theta) = (r, \pi/2)$, $0 \le r \le 1$.

(c) Prove that the solution $G(\boldsymbol{x}, \boldsymbol{x_0})$ you obtained in part (a) satisfies $G(\boldsymbol{x}, \boldsymbol{x_0}) = 0$ for all points Q with polar coordinates $(r,\theta) = (1,\theta)$, $0 \le \theta < \pi/2$.

Hint: you may want to use the formula $d^2 = r^2 + r_0^2 - 2r\, r_0\, \cos(\theta - \theta_0)$ for the distance d between two points of coordinates (r,θ) and (r_0,θ_0).

END