

## EXAMINATION PAPER

Examination Session: May/June

2024

Year:

Exam Code:

MATH2031-WE01

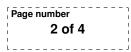
### Title:

# Analysis in Many Variables II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:



#### SECTION A

- Q1 Consider the vector field  $f(x) = x^2 e_1 + \cos y \sin z e_2 + \sin y \cos z e_3$ .
  - (a) By finding a suitable scalar potential, show that f(x) is conservative.
  - (b) Evaluate  $\int_C \boldsymbol{f} \cdot d\boldsymbol{x}$  along the curve C with parametrisation  $\boldsymbol{x}(t) = (t^2 + 1) \boldsymbol{e}_1 + e^t \boldsymbol{e}_2 + e^{2t} \boldsymbol{e}_3, \text{ for } t \in [0, 1].$
- $\mathbf{Q2}$  Compute the surface area of the surface with parametrisation

 $\boldsymbol{x}(u,v) = u \cos v \, \boldsymbol{e}_1 + u \sin v \, \boldsymbol{e}_2 + u^2 \, \boldsymbol{e}_3 \quad \text{for } u \in [0,\sqrt{2}], \ v \in [0,2\pi).$ 

- Q3 (a) Let  $a \in \mathbb{R}$ -{0}. By integrating against an arbitrary test function show that  $(x-a)^2 \delta'(x-a) = 0.$ 
  - (b) Solve the following equation for the generalised function g,

$$(4x^3 - 8x^2 - 3x + 9)g(x) = 0,$$

i.e. find the generalised solution g(x) in terms of shifted delta distributions  $\delta_a$ and possibly their derivatives. Justify the steps taken to arrive at the solution.

Q4 You are given the linear operator

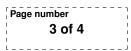
$$L = x^2 e^x \frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x),$$

with the two real-valued functions  $g \in \mathcal{C}^1([1,2])$  and  $h \in \mathcal{C}^0([1,2])$ .

- (a) Calculate the formal adjoint  $L^*$  of L as a function of g and h.
- (b) Choose g so that L is formally self-adjoint.
- (c) Denoting the formally self-adjoint operator found in part (b) as  $\mathfrak{L}$ , consider the Boundary Value Problem on [1, 2] given by

$$\mathfrak{L}u = 0, \qquad u'(1) - u(2) = 0, \ u'(2) = 0.$$

Is this BVP self-adjoint? Justify your answer fully.



### SECTION B

Q5 Consider the vector field

$$f(x) = (x^3 + xz + yz^2) e_1 + (xyz^3 + y^7) e_2 + x^2z^5 e_3,$$

and let S be the union of two smooth surfaces  $S_1$  and  $S_2$ , where  $S_1$  is defined by

$$x^2 + y^2 = 9, \quad z \in [0, 8],$$

and  $S_2$  is defined by

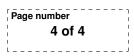
$$x^{2} + y^{2} + (z - 8)^{2} = 9, \quad z \ge 8.$$

- (a) Compute  $\nabla \times \boldsymbol{f}$ .
- (b) Sketch the surface S, and state whether it is open or closed.
- (c) By using Stokes' Theorem, or otherwise, compute the flux of  $\nabla \times \mathbf{f}$  through S, taking the surface normal to be *away from the origin*.

**Q6** A closed surface, S, is given by the parametrisation

$$x(u, v) = \cos u e_1 + \cos v e_2 + \cos(u + v) e_3$$
, for  $u \in [0, \pi], v \in [-\pi, \pi]$ .

- (a) Find a normal vector to S. [This need *not* be a unit vector.]
- (b) Using the Divergence Theorem with the vector field  $f(x) = x e_1$ , calculate the volume enclosed by S.
- Q7 Let  $L = d^2/dx^2$  be the one-dimensional Laplacian.
  - (a) Consider Poisson's equation Lu(x) = f(x) for complex-valued functions  $u \in C^2([0, \ell])$  and periodic boundary conditions  $u(0) = u(\ell)$ . The eigenfunctions for this problem may be expressed in the form  $u_n(x) := \exp(ianx), n \in \mathbb{Z}$  for some real number a.
    - (i) Give the value of  $a \in \mathbb{R}$  in terms of  $\ell$  and give an expression for the normalised eigenfunctions  $\widehat{u_n}(x)$ .
    - (ii) Will Poisson's equation have unique solutions? Why or why not?
    - (iii) Assume that the source term f admits an eigenfunction expansion. Under what conditions (if any) on f would a solution exist?
  - (b) If we instead consider the differential equation Lv(x) = g(x) for functions v on  $[0, \ell]$  satisfying the boundary condition  $v(0) = v(\ell) + 1$ , do these functions v form a vector space? Why or why not?
  - (c) In order to transform the problem of part (b) into the original Lu(x) = f(x)problem with  $u(0) = u(\ell)$  of part (a), one could write u(x) = v(x) + h(x) and f(x) = g(x) + k(x) for some functions h and k. Find the conditions that h and k must satisfy to achieve this goal. Give a simple example of a function h.





 $\mathbf{Q8}$  (a) Consider the two-dimensional domain

$$D = \mathring{D} \cup \ \partial D := \{ (r, \theta) : 0 \le r \le 1, \ 0 \le \theta \le \pi/2 \},$$

where  $\mathring{D} = \{(r,\theta) : 0 < r < 1, 0 < \theta < \pi/2\}$  is the interior of the domain and  $\partial D$  its boundary. Denote the origin of the plane by O and label P the point in  $\mathring{D}$  with  $OP := \mathbf{x}_0 = \frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2$ . Use the method of images to construct the Green's function  $G(\mathbf{x}, \mathbf{x}_0)$  satisfying

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x_0}) = \delta(\boldsymbol{x} - \boldsymbol{x_0}) \qquad \text{for } \boldsymbol{x} \in D,$$
  
$$G(\boldsymbol{x}, \boldsymbol{x_0}) = 0 \qquad \text{for } \boldsymbol{x} \in \partial D.$$

You may use the fact that the fundamental solution of Laplace's equation, which is regular on  $\mathbb{R}^2 - \{x_0\}$ , is given by

$$G_0(\boldsymbol{x}, \boldsymbol{x_0}) = rac{1}{2\pi} \ln |\boldsymbol{x} - \boldsymbol{x_0}|$$

Draw a rough sketch indicating the position of the point P and of its images to support your result for the Green's function  $G(\boldsymbol{x}, \boldsymbol{x_0})$ . Clearly mark the domain D, label your image points as  $P_i$  (with  $OP_i := \boldsymbol{x_i}$ ) and call Q the point such that  $OQ := \boldsymbol{x}$ . Give your answer for the Green's function  $G(\boldsymbol{x}, \boldsymbol{x_0})$ in terms of  $\boldsymbol{x_0}, \boldsymbol{x}$  and  $\boldsymbol{x_i}$ .

- (b) Prove that the solution  $G(\boldsymbol{x}, \boldsymbol{x_0})$  you obtained in part (a) satisfies  $G(\boldsymbol{x}, \boldsymbol{x_0}) = 0$  for all points Q with polar coordinates  $(r, \theta) = (r, \pi/2), 0 \le r \le 1$ .
- (c) Prove that the solution  $G(\boldsymbol{x}, \boldsymbol{x_0})$  you obtained in part (a) satisfies  $G(\boldsymbol{x}, \boldsymbol{x_0}) = 0$  for all points Q with polar coordinates  $(r, \theta) = (1, \theta), \ 0 \le \theta < \pi/2$ .

*Hint*: you may want to use the formula  $d^2 = r^2 + r_0^2 - 2r r_0 \cos(\theta - \theta_0)$  for the distance d between two points of coordinates  $(r, \theta)$  and  $(r_0, \theta_0)$ .