EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH2071-WE01

Title:
Mathematical Physics II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

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SECTION A

Q1 A particle of unit mass is confined to move on the surface of a bowl with shape $z = x^2 + y^2$.

1.1 Write the kinetic energy for the particle, choosing as generalised coordinates r and θ such that

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ z &= r^2 \end{aligned}$$

1.2 Write the Lagrangian, again in terms of r and θ, assuming that the gravitational potential energy is given by gz.

1.3 Find the equations of motion following from the Lagrangian. (You do not need to solve them.)

1.4 Which coordinate is ignorable? Find the (conserved) generalised momentum associated to it.

Q2 Consider a system described by the Hamiltonian

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2$$

2.1 Using the fact that the Poisson bracket

$$\{f, H\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right)$$

of a function $f(q_1, q_2, p_1, p_2)$ of q_i and p_i with the Hamiltonian generates time translations

$$\frac{df}{dt} = \{f, H\}$$

find the expression for the generalised momenta p_i in terms of $\dot{q}_i$.

2.2 Using the expression for p_i you just found, and the relation between the Hamiltonian and the Lagrangian, compute the Lagrangian for the system $L(q_1, q_2, \dot{q}_1, \dot{q}_2)$.

2.3 Consider the infinitesimal transformation

$$\begin{aligned} q_1 \to q_1' &= q_1 + \epsilon q_2 \\ q_2 \to q_2' &= q_2 - \epsilon q_1\,. \end{aligned}$$

Show that this transformation is a symmetry of the Lagrangian you found above.

2.4 The conserved charge that Noether's theorem assigns to the previous symmetry is of the form $Q = q_2 p_1 - p_2 q_1$ (when expressed back in the Hamiltonian formalism). Show by direct computation of the Poisson bracket $\dot{Q} = \{Q, H\}$ that the charge is indeed conserved.

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Q3 Consider a wave function for a particle in a box of size L, so that $0 < x < L$. The wave function is given by

$$\psi(x) = C\sqrt{x^2(L - Dx)}\,.$$

3.1 Determine the constants C and D and motivate your answer.

3.2 Give a sketch of the probability density.

3.3 Compute the expectation value $\langle x \rangle$.

Q4 A quantum mechanical particle in one dimension is subject to a constant force with magnitude F in the positive x-direction.

4.1 Write down the time-independent Schrödinger equation for the wave function $\psi(x)$ of the particle.

4.2 Transform the equation you found above to momentum space.

4.3 Use separation of variables to find the solution $\psi(p)$ up to an overall constant.

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SECTION B

Q5 **5.1** Assume that a field $u(x,t)$ is described by the Lagrangian density

$$\mathcal{L}_0 = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2$$

with $u_t \equiv \partial u/\partial t$ and $u_x \equiv \partial u/\partial t$. Derive the wave equation $u_{tt} = u_{xx}$ from this Lagrangian density, where $u_{tt} \equiv \partial^2 u/\partial t^2$ and $u_{xx} \equiv \partial^2 u/\partial x^2$.

5.2 State D'Alembert's general solution to this wave equation, and use it to find the solution with initial condition

$$u(x,0) = \frac{1}{1+x^2} \qquad ; \qquad u_t(x,0) = 0\,.$$

Sketch how this solution looks for large times.

5.3 Show, using the wave equation you just derived, that energy conservation

$$\frac{dE(a,b)}{dt} = (-u_x u_t)|_{x=a} - (-u_x u_t)|_{x=b}$$

holds for arbitrary solutions $u(x,t)$ of the equations of motion, where $E(a,b)$ is the energy contained in the interval $a \leq x \leq b$:

$$E(a,b) \equiv \int_a^b dx\, \left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right)\,.$$

and $(-u_x u_t)$ encodes the right-moving energy flux.

5.4 More generally, assume that the Lagrangian density is instead

$$\mathcal{L}_m = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}m^2u^2\,.$$

($\mathcal{L}_0$ above is the special case $m = 0$.) Find the equations of motion arising from this Lagrangian density, and use them to show that the expression for the energy flux across the boundary of an interval is the same as before

$$\frac{dE(a,b)}{dt} = (-u_x u_t)|_{x=a} - (-u_x u_t)|_{x=b}$$

but the expression for the energy in the interval gets modified to:

$$E(a,b) \equiv \int_a^b dx\, \left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}m^2u^2\right)\,.$$

5.5 Find the relation between k, m and ω that makes

$$u(x,t) = e^{ikx}e^{i\pm\omega t}$$

a solution of the equations of motion you just derived from $\mathcal{L}_m$.

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Q6 A particle is described by a Lagrangian of the form

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - (\sin(q_1)^2 + \sin(q_2)^2) - \sin(q_1)\sin(q_2)\,.$$

6.1 Find the equations of motion coming from this Lagrangian (you do not need to solve them).

6.2 Show that $q_1 = q_2 = 0$ is a solution of the equations of motion, and write an approximate (second order) Lagrangian describing small oscillations around this solution.

6.3 Find the normal modes for the system, and write down the most general solution of the equations of motion arising from the approximate Lagrangian.

6.4 Assuming that the system starts at rest from $q_1 = q_2 = s$ at $t = 0$, for some small displacement s, find the values of q_1 and q_2 for all t.

Q7 Consider two particles on the real line, described by coordinates x_1 and x_2. Their initial wave function at $t = 0$ is given by

$$\psi(x_1, x_2, t = 0) = \frac{1}{\sqrt{2\pi}} \exp\Big[-\tfrac{1}{4}(x_1)^2 - \tfrac{1}{4}(x_2)^2 \Big] e^{ix_1 - ix_2}\,.$$

7.1 Is this wave function separable?

7.2 Determine the expectation values of the momenta conjugate to x_1 and x_2. Use these to give the interpretation of the wave function $\psi(x_1, x_2, t = 0)$. Motivate your answer.

7.3 At some later time $t > 0$ the wave function is given by

$$\begin{aligned}\psi(x_1, x_2, t = t_1 > 0) \;=\; \frac{C}{\sqrt{2\pi}} \Bigg(&\exp\Big[-\tfrac{1}{4D}(x_1+1)^2 - \tfrac{1}{4D}(x_2-1)^2 \Big] e^{ix_1 - ix_2} \\ &+ \exp\Big[-\tfrac{1}{4D}(x_1-1)^2 - \tfrac{1}{4D}(x_2+1)^2 \Big] e^{ix_2 - ix_1} \Bigg)\,, \qquad (1)\end{aligned}$$

for some constants C and D. Did the particles interact? Do you expect D to be smaller or larger than one? Motivate your answer.

7.4 What is the expectation value $\langle x_2 \rangle$?
Hint: *resist the temptation to compute this by brute force.*

7.5 A measurement is made of the position of particle 1, and one finds $x_1 = 1$. Does this influence the expectation value $\langle x_2 \rangle$? Motivate your answer by referring to the wave function (1).

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Q8 The attractive delta-function potential is given by

$$V(x) = -\alpha\delta(x)\,,$$

where $\alpha > 0$.

8.1 Write down the time-independent Schrödinger equation for a particle in this potential with energy eigenvalue E.

8.2 For a given sign of E, there are two independent solutions to the differential equation for $\psi(x)$ when $x < 0$, and also two independent ones for $x > 0$. Give these 8 solutions. Explain if any of these need to be removed based on physical considerations.

8.3 Which conditions on $\psi(x)$ and $\psi'(x)$ need to be imposed at $x = 0$? Motivate your answer.

8.4 Consider $E < 0$. Express E in terms of the parameter α and the mass m of the particle.

8.5 Now consider $E > 0$ and keep only those solutions for $\psi(x)$ which correspond to either a wave incoming from the left, a reflected wave or a transmitted wave. Show that in the limit $E \to 0$ from above, the potential leads to complete reflection, with no transmission.

END

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