Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH2581-WE01

Title: Algebra II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Let $f(x) = x^5 + \bar{2}x^3 - x^2 + x + \bar{1} \in (\mathbb{Z}/3)[x]$ and $g(x) = x^3 + \bar{2}x^2 + \bar{1} \in (\mathbb{Z}/3)[x]$.

(a) Find the monic $gcd(f(x), g(x))$.

(b) Find $a(x), b(x) \in (\mathbb{Z}/3)[x]$ such that

$$a(x)f(x) + b(x)g(x) = gcd(f(x), g(x)).$$

Q2 Let $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the Gaussian integers, and consider the ideal $I = (2)$.

(a) Show that R/I has exactly four elements: $\bar{0}$, $\bar{1}$, $\bar{i}$ and $\overline{1+i}$.

(b) Give the tables for addition and multiplication in R/I.

(c) Is R/I isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ or $\mathbb{Z}/4$ as a ring? Justify your answer.

Q3 **3.1** Find all groups of the following orders:

(a) 1369, (b) 334.

(Carefully cite any results that you use.)

3.2 Determine all Abelian groups of order 600. Justify your reasoning.

Q4 Consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 4 & 2 & 6 & 5 & 8 \end{pmatrix} \in S_8$.

4.1 Decompose σ as a product of disjoint cycles. Use this to compute σ^2.

4.2 Write σ as a product of transpositions, and determine whether $\sigma \in A_8$.

4.3 Calculate the number of elements conjugate to σ in S_8, and the number of elements conjugate to σ^2 in S_8.

SECTION B

Q5 Let p be a prime number, and $n > 1$ an integer. We write R for the quotient ring $(\mathbb{Z}/p)[x]/(f(x))$, where $f(x) \in (\mathbb{Z}/p)[x]$ is of degree n. Let I be a non-zero ideal in R.

(a) Let $g(x) \in (\mathbb{Z}/p)[x]$ be a monic polynomial of the smallest possible degree such that $\overline{g(x)} \in I$. Show that such a $g(x)$ is unique and that $I = (\overline{g(x)})$.

(b) With notation as above show that $g(x)$ divides $f(x)$ in $(\mathbb{Z}/p)[x]$.

(c) Let $p = 3$, and $f(x) = x^4 - 1$ above. List all ideals in such an R by giving a generator for them.

Q6 (a) List all irreducible polynomials of degree less than or equal to 2 in $(\mathbb{Z}/2)[x]$. Justify your answer.

(b) Show that the polynomial $x^5 + x^2 + \bar{1} \in (\mathbb{Z}/2)[x]$ is irreducible.

(c) Is the polynomial $x^5 + 24x^4 + 56x^3 + 23x^2 + 10x + 11 \in \mathbb{Q}[x]$ irreducible? Justify your answer.

(d) Let F be a field and $f(x), g(x) \in F[x]$, with $g(x)$ a non-zero polynomial. Show that

$$gcd(f(x), g(x)) = gcd(g(x), r(x))$$

where $f(x) = q(x)g(x) + r(x)$ and $gcd(f(x), g(x))$ is the monic gcd of $f(x)$ and $g(x)$ and similarly for $gcd(g(x), r(x))$.

Q7 **7.1** Determine all the group homomorphisms $\phi : \mathbb{Z}/8 \to \mathbb{Z}/12$, and all the isomorphisms $\psi : \mathbb{Z}/6 \to \mathbb{Z}/6$.

7.2 Give a proof or counterexample of the following statement:
If H, K are subgroups of a group G then $H \cup K$ is also a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

7.3 Show that a group G is Abelian if and only if the map $\phi : G \times G \to G$ given by $\phi(a, b) = ab$ is a group homomorphism.

Q8 **8.1** Prove that if H is a subgroup of A_4 with ≥ 7 elements then $H = A_4$. Use this to show that $\langle (1\ 2\ 3), (1\ 2\ 4) \rangle = A_4$.

8.2 Let $n \geq 2$. Let D_{2n} denote the symmetry group of a $2n$-gon. Find a normal subgroup H of D_{2n} such that

$$H \cong D_n \text{ and } D_{2n}/H \cong \mathbb{Z}/2.$$

(For full marks you must justify that it is normal.)

8.3 Find a subgroup in S_5 that is isomorphic to D_5, or prove that such a subgroup does not exist.
(*Hint:* Consider how to choose permutations in S_5 that would yield the appropriate symmetries of a pentagon.)