

EXAMINATION PAPER

Examination Session: May/June

2024

Year:

Exam Code:

MATH2581-WE01

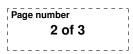
Title:

Algebra II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators
		is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.		

Revision:





SECTION A

Q1 Let
$$f(x) = x^5 + \bar{2}x^3 - x^2 + x + \bar{1} \in (\mathbb{Z}/3)[x]$$
 and $g(x) = x^3 + \bar{2}x^2 + \bar{1} \in (\mathbb{Z}/3)[x]$.

- (a) Find the monic gcd(f(x), g(x)).
- (b) Find $a(x), b(x) \in (\mathbb{Z}/3)[x]$ such that

a(x)f(x) + b(x)g(x) = gcd(f(x), g(x)).

Q2 Let $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the Gaussian integers, and consider the ideal I = (2).

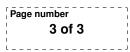
- (a) Show that R/I has exactly four elements: $\overline{0}$, $\overline{1}$, \overline{i} and $\overline{1+i}$.
- (b) Give the tables for addition and multiplication in R/I.
- (c) Is R/I isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ or $\mathbb{Z}/4$ as a ring? Justify your answer.
- Q3 3.1 Find all groups of the following orders:

(Carefully cite any results that you use.)

3.2 Determine all Abelian groups of order 600. Justify your reasoning.

Q4 Consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 4 & 2 & 6 & 5 & 8 \end{pmatrix} \in S_8.$

- **4.1** Decompose σ as a product of disjoint cycles. Use this to compute σ^2 .
- **4.2** Write σ as a product of transpositions, and determine whether $\sigma \in A_8$.
- **4.3** Calculate the number of elements conjugate to σ in S_8 , and the number of elements conjugate to σ^2 in S_8 .



SECTION B

- **Q5** Let p be a prime number, and n > 1 an integer. We write R for the quotient ring $(\mathbb{Z}/p)[x]/(f(x))$, where $f(x) \in (\mathbb{Z}/p)[x]$ is of degree n. Let I be a non-zero ideal in R.
 - (a) Let $\underline{g(x)} \in (\mathbb{Z}/p)[x]$ be a monic polynomial of the smallest possible degree such that $\overline{g(x)} \in I$. Show that such a g(x) is unique and that $I = (\overline{g(x)})$.
 - (b) With notation as above show that g(x) divides f(x) in $(\mathbb{Z}/p)[x]$.
 - (c) Let p = 3, and $f(x) = x^4 1$ above. List all ideals in such an R by giving a generator for them.
- **Q6** (a) List all irreducible polynomials of degree less than or equal to 2 in $(\mathbb{Z}/2)[x]$. Justify your answer.
 - (b) Show that the polynomial $x^5 + x^2 + \overline{1} \in (\mathbb{Z}/2)[x]$ is irreducible.
 - (c) Is the polynomial $x^5 + 24x^4 + 56x^3 + 23x^2 + 10x + 11 \in \mathbb{Q}[x]$ irreducible? Justify your answer.
 - (d) Let F be a field and $f(x), g(x) \in F[x]$, with g(x) a non-zero polynomial. Show that

$$gcd(f(x), g(x)) = gcd(g(x), r(x))$$

where f(x) = q(x)g(x) + r(x) and gcd(f(x), g(x)) is the monic gcd of f(x) and g(x) and similarly for gcd(g(x), r(x)).

- **Q7** 7.1 Determine all the group homomorphisms $\phi : \mathbb{Z}/8 \to \mathbb{Z}/12$, and all the isomorphisms $\psi : \mathbb{Z}/6 \to \mathbb{Z}/6$.
 - **7.2** Give a proof or counterexample of the following statement: If H, K are subgroups of a group G then $H \cup K$ is also a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.
 - **7.3** Show that a group G is Abelian if and only if the map $\phi : G \times G \to G$ given by $\phi(a, b) = ab$ is a group homomorphism.
- **Q8** 8.1 Prove that if *H* is a subgroup of A_4 with ≥ 7 elements then $H = A_4$. Use this to show that $\langle (1 \ 2 \ 3), (1 \ 2 \ 4) \rangle = A_4$.
 - **8.2** Let $n \ge 2$. Let D_{2n} denote the symmetry group of a 2n-gon. Find a normal subgroup H of D_{2n} such that

$$H \cong D_n$$
 and $D_{2n}/H \cong \mathbb{Z}/2$.

(For full marks you must justify that it is normal.)

8.3 Find a subgroup in S_5 that is isomorphic to D_5 , or prove that such a subgroup does not exist.

(*Hint:* Consider how to choose permutations in S_5 that would yield the appropriate symmetries of a pentagon.)