Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH2647-WE01

Title:
Probability II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

SECTION A

Q1 Let X be a random variable with expectation $\mathsf{E}X = a$ and variance $\mathsf{Var}X = \sigma^2 \in (0, \infty)$. For integer $m \geq 1$, define

$$Y_m = \big(X - a\big)^2 \mathsf{1}_{|X-a|>m}.$$

(a) Use Chebyshev's inequality to show that

$$\sup_{m \geq 1} m^2 \mathsf{P}\big(|X - a| > m\big) \leq \sigma^2;$$

(b) By using a suitable limit theorem, find the limit of $\mathsf{E}Y_m$ as $m \to \infty$;

(c) Deduce that

$$\limsup_{m \to \infty} m^2 \mathsf{P}\big(|X - a| > m\big) = 0.$$

In your answer you should clearly state and carefully apply every result you use.

Q2 Let $\big([0,1], \mathcal{B}[0,1], \mathsf{P}\big)$ be the canonical probability space. Given any collection of events $\mathcal{D} \subset \mathcal{B}[0,1]$, write $\sigma(\mathcal{D})$ for the generated σ-field.

Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two collections of events in $\mathcal{B}[0,1]$, where

$$\mathcal{D}_1 = \big\{[a,b] : 0 \leq a < b < 1\big\}, \qquad \mathcal{D}_2 = \big\{[c,d) : 0 \leq c < d \leq 1\big\}.$$

(a) Carefully show that $\mathcal{D}_2 \subset \sigma(\mathcal{D}_1)$;

(b) Carefully show that $\mathcal{D}_1 \subset \sigma(\mathcal{D}_2)$;

(c) Deduce that $\sigma(\mathcal{D}_1) = \sigma(\mathcal{D}_2)$.

In your answer you should properly define generated σ-fields, clearly state and carefully apply every result you use.

SECTION B

Q3 Random bits $(X_n)_{n\geq 1}$ are generated independently with common distribution

$$\mathsf{P}\big(X_n = 0\big) = p, \qquad \mathsf{P}(X_n = 1) = q = 1 - p,$$

where $p \in (0,1)$.

Let T be the first location where the substring 00 is observed,

$$T = \min\big\{n \geq 2 : \, X_{n-1} = X_n = 0\big\}\,.$$

Compute the generating function $\mathsf{E}(s^T)$. Hence, find $\mathsf{P}(T < \infty)$ and $\mathsf{E}(T)$.

Q4 Given $\alpha > 1$ and the canonical probability space $\big([0,1], \mathcal{B}[0,1], \mathsf{P}\big)$, let $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ be sequences of random variables such that

$$\mathsf{P}\big(X_n = n^\alpha\big) = \frac{1}{n} = 1 - \mathsf{P}(X_n = 0)$$

and

$$Y_n(\omega) = n^\alpha \mathsf{1}_{[0,1/n)}(\omega) = \begin{cases} n^\alpha, & \text{if } \omega \in [0, 1/n), \\ 0, & \text{otherwise.} \end{cases}$$

For each of the following claims, determine whether the statement is true or false, and provide a full justification:

(a) As $n \to \infty$, $X_n \to 0$ in L^r for some $r > 1$.
(b) As $n \to \infty$, $X_n \to 0$ in probability.
(c) As $n \to \infty$, $X_n \to 0$ almost surely.
(d) As $n \to \infty$, $Y_n \to 0$ in L^r for some $r > 1$.
(e) As $n \to \infty$, $Y_n \to 0$ in probability.
(f) As $n \to \infty$, $Y_n \to 0$ almost surely.

In your answer you should clearly state and carefully apply any result you use.