Durham University

EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May/June | 2024 | MATH2657-WE01 |

**Title:** Special Relativity and Electromagnetism II

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| Time: | 2 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

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| Instructions to Candidates: | Answer all questions.<br>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.<br>Students must use the mathematics specific answer book. |

| Revision: | |
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## SECTION A

**Q1** **1.1** Rick fires a portal gun that emits green light (wavelength 550 nanometres). Morty is travelling directly towards Rick and assumes that Rick has invented a new portal gun, as it appears to emit blue light (wavelength 450 nanometres).

Calculate the relative speed between Rick and Morty, as a fraction of the speed of light $c$.

Summer is also travelling directly towards Rick, in the same direction that Morty is travelling. The relative speed between Rick and Summer is $c/2$.

Calculate the relative speed between Summer and Morty, as a fraction of the speed of light $c$.

**1.2** Two particles, each of rest mass $m$, collide and fuse to form a particle with rest mass $6m$.

Find the speeds of the two initial particles, as a fraction of the speed of light $c$, in the rest frame of the final particle.

The final particle subsequently decays into a pair of identical particles, each with a kinetic energy that is equal to a quarter of its rest mass.

Find the speeds of the particles produced by this decay, as a fraction of the speed of light $c$.

**Q2** **2.1** The electric charge density vanishes outside a ball of radius $a$ centred at the origin. Inside the ball it is given by

$$\rho = \frac{q(\beta a - r)}{a^4},$$

where $q, \beta$ are positive constants and $r$ is the distance to the centre of the ball.

Calculate the total charge inside the ball, in terms of $q$ and $\beta$.

Calculate the value of $\beta$ so that a point particle with charge $5q$ placed at a distance $7a$ from the centre of the ball feels no force due to the ball.

**2.2** State the transformation law for the magnetic vector potential $\mathbf{A}$ under a gauge transformation and use this to obtain a Poisson equation that can be solved to obtain the magnetic vector potential in Coulomb gauge.

Calculate the magnetic field $\mathbf{B}$, from the magnetic vector potential

$$\mathbf{A} = \lambda(kxyz, y^2z, yz^2),$$

where $\lambda$ and $k$ are constants, and use the result to explain why varying the value of $k$ does not correspond to a gauge transformation.

Find the value of $k$ so that this magnetic vector potential is in Coulomb gauge.

## SECTION B

**Q3** (a) The frame $\mathcal{R}'$ is obtained from the frame $\mathcal{R}$ by applying a Lorentz boost along the positive $x$-axis. This Lorentz boost is given by a matrix of the form

$$L_1 = \begin{bmatrix} \frac{5}{3} & -a_1 & 0 & 0 \\ -a_1 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Determine the value of the positive constant $a_1$ and the speed of the Lorentz boost, as a fraction of the speed of light $c$.

(b) The frame $\mathcal{R}''$ is obtained from the frame $\mathcal{R}'$ by applying a Lorentz boost along the positive $y'$-axis, with the same speed as the boost in part (a). Give the matrix $L_2$ for this Lorentz boost.

(c) Verify that the frame $\mathcal{R}''$ is obtained from the frame $\mathcal{R}$ by applying a Lorentz transformation of the form

$$L_3 = \frac{1}{9}\begin{bmatrix} 25 & -20 & -12 & 0 \\ -12 & 15 & 0 & 0 \\ -20 & a_3 & 15 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix},$$

where $a_3$ is a positive constant that you should determine.

(d) The Lorentz transformation $L_3$ from part (c) may be represented as the Lorentz boost $L_4$ followed by a rotation around the $z''$-axis by an angle $\theta$, where $L_4$ has the form

$$L_4 = \frac{1}{153}\begin{bmatrix} 425 & -340 & -204 & 0 \\ -340 & 353 & a_4 & 0 \\ -204 & a_4 & 225 & 0 \\ 0 & 0 & 0 & 153 \end{bmatrix}.$$

Find the values of the positive constants $a_4$ and $\cos\theta$.
[Hint: You may find it useful to recall that $204 = 17 \times 12$ and $153 = 17 \times 9$]

(e) Calculate the speed $v_4$, as a fraction of the speed of light $c$, of the Lorentz boost $L_4$ given in part (d).

(f) State a property that is satisfied by a matrix describing a Lorentz boost, that is not satisfied by all matrices that describe Lorentz transformations, and that can be used to conclude that $L_3$ is not a Lorentz boost. Explain why this property suggests that a Lorentz boost followed by another Lorentz boost is generally not expected to be a Lorentz boost. State a restriction under which a Lorentz boost followed by another Lorentz boost is equal to a Lorentz boost.

**Q4** (a) In electrostatics, Maxwell's equations reduce to two remaining equations for the electric field $\mathbf{E}$. State these two equations.

(b) Consider the special case that the electric charge density $\rho$ is a function of only the combination $R = \sqrt{x^2 + y^2}$. Show that by taking the electric field to have the form $\mathbf{E} = (x, y, 0)f$, where $f$ is a function of $R$ only, then one of the equations from part (a) is solved automatically and the other becomes an ordinary differential equation for $f$.

(c) Use the form of the electric field given in part (b) and the integral form of Gauss' law to calculate the function $f(R)$ due to the electric charge density

$$\rho = \alpha e^{-\beta(x^2+y^2)},$$

where $\alpha$ and $\beta$ are positive constants.

(d) Verify that the function $f(R)$ obtained in part (c) solves the ordinary differential equation obtained in part (b) for $R > 0$.