EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH2707-WE01

Title:
Markov Chains II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2024

SECTION A

Q1 Let

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

(a) Compute all possible stationary distributions for P.

(b) Determine the period of each state.

You must justify your solutions.

Q2 Consider a rook moving uniformly at random from amongst the allowed moves on the unusually shaped chessboard depicted in Figure 1. If the rook starts on the square labelled x, what is the expected return time of the rook? You must justify your solution.

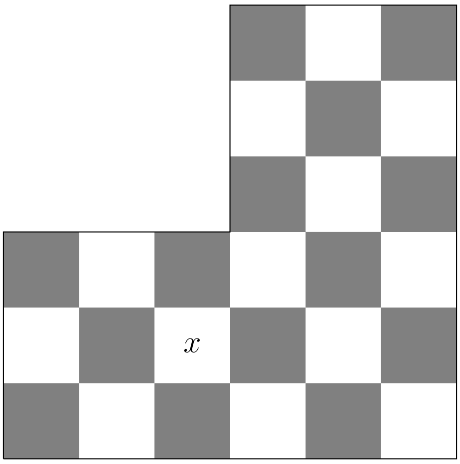

Figure 1: The unusually shaped chessboard for problem **Q2**. A rook can move any positive number of squares in a straight line (horizontally or vertically). For example, there are seven possible places to move to from the square labelled x.

SECTION B

Q3 Let $I = \mathbb{Z}$, the set of all integers. Let $\bar{P}$ be the stochastic matrix indexed by I with

$$\bar{p}_{ij} = \begin{cases} \frac{1}{2} & |i-j| = 1, \\ 0 & \text{else}, \end{cases}$$

and for $m = 1, 2, 3$ let $Q^{(m)}$ denote the matrices with entries

$$Q^{(1)}_{ij} = \begin{cases} 1 & j = 0, \\ 0 & \text{else}, \end{cases} \qquad Q^{(2)}_{ij} = \begin{cases} 1 & i \text{ odd}, j = 0, \\ \frac{1}{2} & i \text{ even}, |i-j| = 1, \\ 0 & \text{else}, \end{cases}$$

and

$$Q^{(3)}_{ij} = \begin{cases} 1 & i = 4k+1 \text{ for some } k \in I, j = 0, \\ \frac{1}{2} & i \neq 4k+1 \text{ for any } k \in I, |i-j| = 1, \\ 0 & \text{else}. \end{cases}$$

Let $P^{(m)}$ denote the stochastic matrix $\frac{1}{2}\bar{P} + \frac{1}{2}Q^{(m)}$.

(a) Is the state 0 recurrent or transient for a Markov chain started at the state 0 and with transition matrix $P^{(1)}$?

(b) Is the state 0 recurrent or transient for a Markov chain started at the state 0 and with transition matrix $P^{(2)}$?

(c) Is the state 0 recurrent or transient for a Markov chain started at the state 0 and with transition matrix $P^{(3)}$?

You must justify your solutions.

Q4 Let $I = \{1, 2, 3, \dots\}$. For $n \in I$ let $a_n = n^{-2}$. In this problem you may take for granted the fact that $\sum_{n=1}^{\infty} a_n = \pi^2/6$.

(a) Find a transition matrix P indexed by I and a constant $c > 0$ such that both

  (i) $p_{ij} = 0$ if $|i-j| > 1$ and

  (ii) the stationary distribution σ of P satisfies $\sigma_n = c a_n$ for all $n \in I$.

(b) Let $(X_n)_{n\geq 0}$ be a Markov chain with initial state 1 and a transition matrix P that satisfies the requirements in part (a). Is this Markov chain recurrent or transient?

(c) Let A denote the long-run fraction of time that $(X_n)_{n\geq 0}$ spends at the state 1. Compute $\cos(A\pi^3)$.

You must justify your solutions.