EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH2727-WE01

Title:

Topology II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 (a) Let X be a topological space. Define what it means for X to be connected. Give an alternative (and equivalent) characterisation of connectedness.

(b) Show the equivalence of the two characterisations given in your answer to the previous item.

(c) Suppose X and Y are topological spaces, X is connected, and $f : X \to Y$ is a continuous map. Show that $f(X)$ is connected.

Q2 Suppose we are given a set X and two topologies τ, τ' on X. We say that τ is **finer** than τ' if $\tau \supseteq \tau'$.

(a) Show that the identity

$$\mathrm{id}\colon (X, \tau) \to (X, \tau')$$

is continuous (as a map from X equipped with τ to X equipped with τ') if and only if τ is finer than τ'.

(b) Suppose (X, τ) is compact and (X, τ') is Hausdorff. In addition, suppose that τ is finer than τ'. Show that $\tau = \tau'$.

(c) In part (b), can we omit the assumption of Hausdorffness on (X, τ')? In other words, suppose (X, τ) is compact and τ is finer than τ'. Is it always the case that $\tau = \tau'$? Justify your answer.

SECTION B

Q3 Let X be a set. We call a function $\rho\colon X \times X \to [0, \infty)$ a **pseudometric** (and the pair (X, ρ) a **pseudometric space**) if for all $x, y, z \in X$ the following three assumptions hold

- $\rho(x, x) = 0$;
- $\rho(x, y) = \rho(y, x)$;
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

NB.: Note the slight difference to the definition of a metric in the first item.

(a) Let (X, ρ) be a pseudometric space. For each $x \in X$ and $\varepsilon > 0$, set

$$B_\varepsilon(x) := \{y \in X\colon \rho(x, y) < \varepsilon\}.$$

Define

$$\tau := \{U \subseteq X\colon \text{ for each } x \in U \text{ there is } \varepsilon > 0 \text{ with } B_\varepsilon(x) \subseteq U\}.$$

Show that τ is a topology on X. Below, we may call τ **induced** by ρ.

(b) Consider the topology τ from item (a). Give a necessary and sufficient criterion (in terms of assumptions on ρ) for τ to be Hausdorff. Justify your answer.

(c) Given a topological space (X, τ), we may call $x, y \in X$ **indistinguishable** if we have for each $U \in \tau$ that $x \in U$ if and only if $y \in U$. Show that if τ is induced by a pseudometric, then x, y are indistinguishable if and only if $\rho(x, y) = 0$.

(d) Let $X := \mathbb{R} \cup \{\overline{0}\}$ be the real line with the origin doubled. For $\varepsilon > 0$, set $U_\varepsilon := (-\varepsilon, 0) \cup \{\overline{0}\} \cup (0, \varepsilon) \subseteq X$. In the lectures, we defined a topology τ on X which can equivalently be characterised via the basis

$$\mathcal{B} = \{U \subseteq X \colon U = (a, b) \subseteq \mathbb{R} \text{ for some } a, b \in \mathbb{R} \text{ or } U = U_\varepsilon \text{ for some } \varepsilon > 0\}.$$

Show that τ is not induced by a pseudometric.

Hint: Are there indistinguishable points in the real line with the origin doubled?

Q4 In the following, consider

$$Y = \left\{ \left(x, \sin\frac{1}{x}\right) \in \mathbb{R}^2 \,|\, 0 < x < \infty \right\},$$

$Z = \{(0, y) \in \mathbb{R}^2 \,|\, y \in [-1, 1]\}$, and $X = Y \cup Z$ equipped with the subspace topology induced by the standard topology on $\mathbb{R}^2$.

(a) Show that Y is homeomorphic to $(0, \infty)$.

(b) Show that $X = \overline{Y}$.

(c) Show that X/Z is homeomorphic to $[0, \infty)$.