

## **EXAMINATION PAPER**

Examination Session: May/June

2024

Year:

Exam Code:

MATH3011-WE01

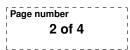
Title:

Analysis III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Materials i erifilited.		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.	

**Revision:** 



## SECTION A

- **Q1** Throughout this question, E is assumed to be a subset of the real numbers  $\mathbb{R}$ .
  - 1.1 (a) State what it means for E to be countable.(b) Give an example of a countable set E.
  - **1.2** Show that if E is countable then E has Lebesgue outer measure equal to 0.
  - **1.3** (a) State what it means for E to be Lebesgue measurable.
    - (b) Show that if E is countable then E is Lebesgue measurable.
    - (c) Give an example of a Lebesgue measurable E that is uncountable.
- **Q2** 2.1 Let  $g : \mathbb{R} \to (0, \infty)$  and  $f : \mathbb{R} \to \mathbb{R}$ . Assume that g and f are (Lebesgue) measurable.
  - (a) State the definition of the integral  $\int g$ . You may assume the definition of the integral of a nonnegative simple measurable function.
  - (b) State what it means for f to be integrable and state the definition of the integral  $\int f$ .
  - **2.2** Show that the function  $f : \mathbb{R} \to \mathbb{R}$  given by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \in (0,1) \cap \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

is (Lebesgue) measurable and integrable.

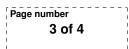
Q3 3.1 Let  $E \subset \mathbb{R}$  be measurable and let  $1 \leq p < \infty$ . State the definition of  $L^p(E)$ . 3.2 Consider  $f : \mathbb{R} \to \mathbb{R}, f \in L^2(\mathbb{R})$ . For  $n \in \mathbb{N}$ , we define

$$g_n(x) := \begin{cases} f(x), & \text{if } 0 \le f(x) \le n, \\ n, & \text{if } f(x) > n. \end{cases}$$

- (a) Prove that  $g_n \in L^2(\mathbb{R})$ .
- (b) Does  $(g_n)_n$  converge to f in  $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$ ? Give a full justification of your response.
- **Q4** 4.1 Let  $(X, \|\cdot\|)$  be a normed linear space. State the definition of a bounded linear functional  $T: X \to \mathbb{R}$ .
  - **4.2** Let  $C^{1}[0,1]$  be the normed linear space of real-valued differentiable functions on [0,1] with norm

$$||f|| = \max_{x \in [0,1]} |f(x)|, \quad f \in C^1[0,1].$$

- (a) Prove that  $T_1: C^1[0,1] \to \mathbb{R}$  defined by  $T_1(f) := \int_0^1 f(x) dx$  is a bounded linear functional on  $C^1[0,1]$ .
- (b) Give an example of a linear functional  $T_2 : C^1[0,1] \to \mathbb{R}$  which is not bounded. Provide a full justification of your response.



## SECTION B

- Q5 Recall that  $\mathcal{M}$  denotes the measurable functions with domain  $\mathbb{R}$  and taking values in the extended reals; and that  $\mathcal{M}^+$  denotes the measurable functions with domain  $\mathbb{R}$  and taking values in the nonnegative extended reals.
  - **5.1** Let  $f, h \in \mathcal{M}$ . Suppose that h is integrable.
    - (a) Suppose that  $|f| \leq h$ . Explain why it follows that f is integrable.
    - (b) Show that h is finite almost everywhere.
  - **5.2** State Fatou's Lemma. (*Hint: this is a statement concerning*  $\mathcal{M}^+$ .)
  - **5.3** Let  $f_n \in \mathcal{M}$ , for  $n \in \mathbb{N}$ . Suppose that  $h \in \mathcal{M}$  is integrable and that  $|f_n| \leq h$  for every  $n \in \mathbb{N}$ . In the following, you may assume that  $\liminf_{n\to\infty} f_n \in \mathcal{M}$ .
    - (a) Explain why  $\liminf_{n\to\infty} f_n$  is integrable.
    - (b) Show that  $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$ .

Give full justifications of your responses.

**Q6** 6.1 Let  $f : \mathbb{R} \to \mathbb{R}$  and denote

$$f^+: \mathbb{R} \to \mathbb{R}$$

$$f^+(x) = \max\{0, f(x)\}.$$

- (a) State what it means for f to be (Lebesgue) measurable.
- (b) Show that if f is (Lebesgue) measurable then the function  $f^+$  is (Lebesgue) measurable. Fully justify your answer from the definition of measurability.
- **6.2** Let C be the middle-third Cantor set. Define the function  $h: [0,1] \to \mathbb{R}$  by

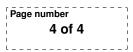
$$h(x) = \begin{cases} \sin(1/x) & \text{if } x \in [0,1] \cap C \\ 0 & \text{if } x \notin C. \end{cases}$$

- (a) Show that h is continuous on a set E with [0,1] E having Lebesgue measure 0.
- (b) Is the function h Riemann integrable over [0,1]? Briefly justify your answer.
- **6.3** Assume that  $r : \mathbb{R} \to \mathbb{R}$  is a function for which

$$2^{-1}|x-y| \le |r(x) - r(y)| \le 2|x-y|,$$

for all  $x, y \in \mathbb{R}$ . In the following, you may use the fact that for any interval I,  $r^{-1}(I)$  is an interval with length at most  $2\ell(I)$ .

- (a) Show that for any  $E \subset \mathbb{R}$  we have  $\mu^*(r^{-1}(E)) \leq 2\mu^*(E)$ .
- (b) Show that if E is Lebesgue measurable then  $r^{-1}(E)$  is Lebesgue measurable. able. You may use that a function  $f : \mathbb{R} \to \mathbb{R}$  is continuous if and only if for all open sets  $U \subset \mathbb{R}$ ,  $f^{-1}(U) \subset \mathbb{R}$  is open.
- (c) Show that for any  $f \in \mathcal{M}^+$  we have  $\int f \circ r \leq 2 \int f$ . You may use the identity  $\mathbb{1}_E \circ r = \mathbb{1}_{r^{-1}(E)}$ .





**Q7** Let  $[a, b] \subset \mathbb{R}$  be a closed bounded interval. For  $f \in L^1[a, b]$ , consider

$$||f|| = \int_{[a,b]} x^2 |f(x)|.$$

- **7.1** Is  $\|\cdot\|$  a norm on  $L^1[a, b]$ ? Justify your response.
- **7.2** Is  $(L^1[a, b], \|\cdot\|)$  a Banach space? Justify your response.
- **7.3** Give an example of a normed linear space  $(X, \|\cdot\|)$  such that there is a sequence of functions that is Cauchy in  $(X, \|\cdot\|)$  but does not converge in  $(X, \|\cdot\|)$ . Briefly justify your response.
- **Q8** 8.1 Let *H* be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and let  $\|\cdot\|$  denote the norm derived from the inner product. Let *V* be a finite dimensional subspace of *H*.
  - (a) State what it means for  $\{\varphi_1, \ldots, \varphi_n\}$  to be an orthonormal set in  $(V, \langle \cdot, \cdot \rangle)$ .
  - (b) Now suppose  $\{\varphi_1, \ldots, \varphi_n\}$  is an orthonormal basis of  $(V, \langle \cdot, \cdot \rangle)$ . Show that for each  $u \in H$ , there exists a  $w \in V$  such that

$$||u - w|| = \min_{v \in V} ||u - v||$$

by deriving an explicit formula for w.

**8.2** Let G, F be infinite dimensional, separable Hilbert spaces. Prove that there exists a bijective linear transformation  $T: F \to G$  such that

$$||T(h)||_G = ||h||_F,$$

where  $\|\cdot\|_G$ ,  $\|\cdot\|_F$  are the norms derived from the inner products on G, F respectively. You may use without proof that any separable Hilbert space has a countable orthonormal basis.