EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3021-WE01

Title:
Differential Geometry III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Let $\alpha : \mathbb{R} \to \mathbb{R}^3$,

$$\alpha(t) = \left(t, \frac{t^2}{2}, \frac{t^3}{3}\right).$$

Calculate the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the curve α for arbitrary $t \in \mathbb{R}$ and for $t = 1$.

Q2 Let $\mathbb{H} = \{(u, v) \in \mathbb{R}^2 : v > 0\}$ be the hyperbolic plane with first fundamental form

$$E(u, v) = G(u, v) = \frac{1}{v^2} \quad \text{and} \quad F(u, v) = 0.$$

(a) Let $\alpha : [0, 1] \to \mathbb{H}$ be given by $\alpha(t) = (t, 1 + ct)$ for a fixed constant $c > 0$. Derive the hyperbolic length of the curve α.

(b) Find the hyperbolic area of the domain

$$A_{a,b,c} := \{(u, v) \in \mathbb{H} : a \leq u \leq b, v \geq c\}$$

for fixed $c > 0$ and $a < b$. Use this result to decide whether the domain

$$D := \{(u, v) \in \mathbb{H} : u \geq 1, v \geq u^2\}$$

has finite or infinite hyperbolic area.

Q3 Let $S \subset \mathbb{R}^3$ be a regular surface.

(a) Give the definition of a geodesic $c : I \to S$, where $I \subset \mathbb{R}$ is an interval.

(b) Let $c : \mathbb{R} \to S$ be a geodesic with $c'(t) \neq 0$ for all $t \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}, t = f(r)$ be a smooth invertible function with smooth inverse $r = f^{-1}(t)$. Show that $c \circ f^{-1}(t)$ is again a geodesic in S if and only if f is a linear function.

Q4 Let $\mathbb{H}^2 = \{(u, v) \in \mathbb{R}^2 : v > 0\}$ be the hyperbolic plane with $E(u, v) = G(u, v) = \frac{1}{v^2}$ and $F(u, v) = 0$. Let $A \subset \mathbb{H}^2$ be the bounded domain (with respect to the hyperbolic metric) bounded by the four curves $u = -1/2$, $u = 1/2$, $u^2+v^2 = 1/2$ and $u^2+v^2 = 1$ with the four vertices $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ and interior angles $\pi/4, \pi/4, 2\pi/3, 2\pi/3$ at these vertices (you do not need to prove this). Let $B \subset \mathbb{H}^2$ be the Euclidean rectangle with these four vertices.

(a) Derive the hyperbolic area of the domain A by applying the Gauss-Bonnet Theorem.

(b) Prove or disprove that the hyperbolic area of the domain B is nonvanishing.

SECTION B

Q5 (a) Let $\alpha : I \to \mathbb{R}^2$ be a smooth unit speed plane curve. Define the curvature function $\kappa : I \to \mathbb{R}$ of α and give the moving frame equations.

(b) Let $\alpha, \beta : I \to \mathbb{R}^2$ be two smooth unit speed plane curves, where $\alpha(s_0) = \beta(s_0)$ and $\alpha'(s_0) = \beta'(s_0)$ for some fixed $s_0 \in I$. Assume that the curvature function of α agrees with the curvature function of β. Prove that $\alpha = \beta$ by using the function

$$f(s) = \|t_\beta(s) - t_\alpha(s)\|^2 + \|n_\beta(s) - n_\alpha(s)\|^2,$$

where $t_\alpha, n_\alpha : I \to \mathbb{R}^2$ and $t_\beta, n_\beta : I \to \mathbb{R}^2$ are the unit tangent and unit normal vector functions of α and β, respectively.

Q6 (a) Let $S \subset \mathbb{R}^3$ be a regular surface and $\alpha, \beta : [a, b] \to S$ be two regular curves with $\alpha(s) = \beta(t) \in S$. Give a formula for the cosine of the angle between them at this intersection point.

(b) Let $x(u, v)$ be a parametrisation of a surface $S \subset \mathbb{R}^3$ with the following coefficients of the first fundamental form:

$$E(u, v) = 1 + 4u^2, \quad F(u, v) = \frac{4}{3}uv, \quad G(u, v) = 1 + \frac{4}{9}v^2.$$

Let $\alpha : [0, 2\pi] \to S$ and $\beta : [0, \infty) \to S$ be the curves

$$\alpha(s) = x(\cos s, \sin s), \quad \beta(t) = x(t, t\sqrt{3}).$$

Show that we have for the angle of intersection θ of both curves:

$$\cos\theta = -\frac{1}{2\sqrt{2}}.$$

Q7 Let $S \subset \mathbb{R}^3$ be a regular surface.

(a) Let $c : I \to S$ be a smooth unit speed curve with $I \subset \mathbb{R}$ an interval. Give the definition of the geodesic curvature and normal curvature of c.

(b) Give the definition of an asymptotic curve $\alpha : I \to S$ in S, where $I \subset \mathbb{R}$ is an interval.

(c) Give and justify a condition which ensures a regular curve is an asymptotic curve. You may quote Theorems from class in doing so.

Q8 (a) Let $S \subset \mathbb{R}^3$ be a regular surface which is globally parametrised by $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2 \right).$$

Calculate the coefficients E, F, G of the first fundamental form with respect to $\mathbf{x}$ and prove or disprove that $\mathbf{x}$ is isothermal.

(b) Prove or disprove that the coefficients L, M, N of the second fundamental form with respect to $\mathbf{x}$ in (a) are given by $L = 2, M = 0, N = -2$.

(c) Calculate the principal curvatures of the surface S in (a).

END