

EXAMINATION PAPER

Examination Session: May/June

2024

Year:

Exam Code:

MATH3041-WE01

Title:

Galois Theory III

| Time: | 3 hours | |
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| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

| Instructions to Candidates: | Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book. | |
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Revision:



SECTION A

Q1 1.1 Given a finite degree field extension L/K, state the definition of the minimal polynomial of an element $\alpha \in L$ over K. How many elements in the finite field \mathbb{F}_{128} have a minimal polynomial of degree

5 over \mathbb{F}_2 ? 2 Given a field K and natural number n, state the definition of a primitive n-th

1.2 Given a field K and natural number n, state the definition of a primitive n-th root of unity in K.

Are there any primitive 4-th roots of unity in the finite field \mathbb{F}_{343} ?

Q2 2.1 Given a finite degree field extension L/K, state what it means for the extension to be normal.

Given an intermediate field $K \subset M \subset L$, prove that if L/K is normal, then so is L/M.

2.2 Find all the roots of the following quartic polynomial

$$X^4 + \frac{7}{2}X^2 - \sqrt{15}X + \frac{21}{16} \in \mathbb{C}[X].$$

- **Q3** (a) Given $\theta = \sqrt[3]{3} + \sqrt[3]{9}$, show that $\mathbb{Q}(\theta) = \mathbb{Q}(\sqrt[3]{3})$.
 - (b) Find the minimal polynomial of $\theta = \sqrt[3]{3} + \sqrt[3]{9}$ over \mathbb{Q} .
 - (c) Find the minimal polynomial of $\theta = \sqrt[3]{3} + \sqrt[3]{9}$ over $\mathbb{Q}(\sqrt{3})$.
- **Q4** (a) Find the Galois group of $X^3 3X 1$ over \mathbb{Q} .
 - (b) Find the Galois group of $X^3 3X 2$ over \mathbb{Q} .
 - (c) Find the Galois group of $X^3 3X 3$ over \mathbb{Q} .

SECTION B

- **Q5** 5.1 Let $\mathbb{C}(X)$ be the field of rational functions with indeterminate X and coefficients in \mathbb{C} . Suppose that $Y = X^2 X^{-2}$.
 - (a) Prove that $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension.
 - (b) Apply Galois theory to list all fields between $\mathbb{C}(Y)$ and $\mathbb{C}(X)$.
 - **5.2** Find an irreducible polynomial of degree 12 in $\mathbb{F}_2[X]$.

Q6 Let $L = \mathbb{Q}(\zeta)$ where $\zeta = \exp(2\pi i/120) \in \mathbb{C}$ is a primitive 120-th root of unity.

- (a) Describe the structure of $\operatorname{Gal}(L/\mathbb{Q})$ as a product of cyclic groups.
- (b) Apply Galois theory to list all the distinct intermediate fields $\mathbb{Q} \subset M \subset L$ with $[M : \mathbb{Q}] = 2$.
- (c) Calculate the number of distinct intermediate fields $\mathbb{Q} \subset N \subset L$ such that the Galois group $\operatorname{Gal}(L/N)$ is isomorphic to $\mathbb{Z}/4$. (You do not need to list these fields.)

Q7 Let *L* be a splitting field for the polynomial $X^4 + 6X^2 - 10$ over \mathbb{Q} .

- (a) Describe the group structure of $\operatorname{Gal}(L/\mathbb{Q})$.
- (b) List all subfields $M \subset L$ such that M/\mathbb{Q} is Galois.
- (c) For each above subfield $M \subset L$, describe the structure of the group $\operatorname{Gal}(L/M)$.

Q8 Let $\zeta = \exp(2\pi i/32) \in \mathbb{C}$ and set $L = \mathbb{Q}(\theta)$ where $\theta = \zeta + \zeta^{-1} = 2\cos(\pi/16)$.

- (a) Prove that L/\mathbb{Q} is a cyclic field extension of degree 8. Find the minimal polynomial of ζ over L. Also, express the eight Galois conjugates of θ in the extension L/\mathbb{Q} in the form $2\cos(t)$ for suitable values of $t \in [0, \pi]$.
- (b) Prove that $L_1 = \mathbb{Q}(2\cos(\pi/8)), L_2 = \mathbb{Q}(2\cos(\pi/4))$ and $L_3 = \mathbb{Q}$ are the only proper subfields of L. Find the minimal polynomials of θ over L_i for each i = 1, 2, 3. You may find the identity $2\cos(2t) = (2\cos t)^2 2$ useful.
- (c) Prove that $\theta = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$. Is it true that $\theta' = \sqrt{2 + \sqrt{2 \sqrt{2}}}$ is one of conjugates of θ from part (a)?