Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH30420-WE01

Title:
Galois Theory V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 **1.1** Given a finite degree field extension L/K, state the definition of the minimal polynomial of an element $\alpha \in L$ over K.

How many elements in the finite field $\mathbb{F}_{128}$ have a minimal polynomial of degree 5 over $\mathbb{F}_2$?

1.2 Given a field K and natural number n, state the definition of a primitive n-th root of unity in K.

Are there any primitive 4-th roots of unity in the finite field $\mathbb{F}_{343}$?

Q2 **2.1** Given a finite degree field extension L/K, state what it means for the extension to be normal.

Given an intermediate field $K \subset M \subset L$, prove that if L/K is normal, then so is L/M.

2.2 Find all the roots of the following quartic polynomial

$$X^4 + \frac{7}{2}X^2 - \sqrt{15}X + \frac{21}{16} \in \mathbb{C}[X].$$

Q3 (a) Given $\theta = \sqrt[3]{3} + \sqrt[3]{9}$, show that $\mathbb{Q}(\theta) = \mathbb{Q}(\sqrt[3]{3})$.

(b) Find the minimal polynomial of $\theta = \sqrt[3]{3} + \sqrt[3]{9}$ over $\mathbb{Q}$.

(c) Find the minimal polynomial of $\theta = \sqrt[3]{3} + \sqrt[3]{9}$ over $\mathbb{Q}(\sqrt{3})$.

Q4 (a) Find the Galois group of $X^3 - 3X - 1$ over $\mathbb{Q}$.

(b) Find the Galois group of $X^3 - 3X - 2$ over $\mathbb{Q}$.

(c) Find the Galois group of $X^3 - 3X - 3$ over $\mathbb{Q}$.

SECTION B

Q5 **5.1** Let $\mathbb{C}(X)$ be the field of rational functions with indeterminate X and coefficients in $\mathbb{C}$. Suppose that $Y = X^2 - X^{-2}$.

(a) Prove that $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension.

(b) Apply Galois theory to list all fields between $\mathbb{C}(Y)$ and $\mathbb{C}(X)$.

5.2 Find an irreducible polynomial of degree 12 in $\mathbb{F}_2[X]$.

Q6 Let $L = \mathbb{Q}(\zeta)$ where $\zeta = \exp(2\pi i/120) \in \mathbb{C}$ is a primitive 120-th root of unity.

(a) Describe the structure of $\mathrm{Gal}(L/\mathbb{Q})$ as a product of cyclic groups.

(b) Apply Galois theory to list all the distinct intermediate fields $\mathbb{Q} \subset M \subset L$ with $[M : \mathbb{Q}] = 2$.

(c) Calculate the number of distinct intermediate fields $\mathbb{Q} \subset N \subset L$ such that the Galois group $\mathrm{Gal}(L/N)$ is isomorphic to $\mathbb{Z}/4$.

(You do not need to list these fields.)

Q7 Let L be a splitting field for the polynomial $X^4 + 6X^2 - 10$ over $\mathbb{Q}$.

(a) Describe the group structure of $\mathrm{Gal}(L/\mathbb{Q})$.

(b) List all subfields $M \subset L$ such that $M/\mathbb{Q}$ is Galois.

(c) For each above subfield $M \subset L$, describe the structure of the group $\mathrm{Gal}(L/M)$.

Q8 Let $\zeta = \exp(2\pi i/32) \in \mathbb{C}$ and set $L = \mathbb{Q}(\theta)$ where $\theta = \zeta + \zeta^{-1} = 2\cos(\pi/16)$.

(a) Prove that $L/\mathbb{Q}$ is a cyclic field extension of degree 8. Find the minimal polynomial of ζ over L. Also, express the eight Galois conjugates of θ in the extension $L/\mathbb{Q}$ in the form $2\cos(t)$ for suitable values of $t \in [0, \pi]$.

(b) Prove that $L_1 = \mathbb{Q}(2\cos(\pi/8))$, $L_2 = \mathbb{Q}(2\cos(\pi/4))$ and $L_3 = \mathbb{Q}$ are the only proper subfields of L. Find the minimal polynomials of θ over L_i for each $i = 1, 2, 3$. *You may find the identity* $2\cos(2t) = (2\cos t)^2 - 2$ *useful.*

(c) Prove that $\theta = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$. Is it true that $\theta' = \sqrt{2 + \sqrt{2 - \sqrt{2}}}$ is one of conjugates of θ from part (a)?